

# 6.300: Signal Processing

Tutorial Topic

Circular Convolution

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## Circular Convolution

Multiplication of  $N$ -point DFTs corresponds to time-domain circular convolution.

$$\frac{1}{N}(x \circledast h)[n] \longleftrightarrow X_N[k]H_N[k] \quad (x \circledast h)[n] = N \text{DFT}_N^{-1}\{X_N[k]H_N[k]\}$$

$$(x \circledast h)[n] = \sum_{m=0}^{N-1} x[m]h[(n-m) \bmod N] = (h \circledast x)[n] = \sum_{m=0}^{N-1} h[m]x[(n-m) \bmod N]$$

Computing a circular convolution may seem complicated, but it is really simple.

- Perform linear convolution — the convolution you're already familiar with.
- Wrap the result of linear convolution into a length- $N$  interval.  
This is *time-aliasing*, which is the time-domain analogue of the more familiar frequency-aliasing.
- Periodically extend every  $N$  samples.

Compute the result using superposition. Use a table or sketch a plot to keep track of the values.

|        |   |   |   |   |   |   |   |   |   |   |   |
|--------|---|---|---|---|---|---|---|---|---|---|---|
| $n$    | = | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $x[n]$ | = | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $h[n]$ | = | 1 | 2 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

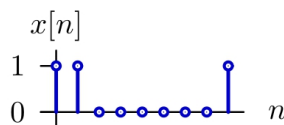
|                          |   |    |    |    |    |    |    |    |    |    |    |
|--------------------------|---|----|----|----|----|----|----|----|----|----|----|
| $n$                      | = | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| $h[0]x[n-0]$             | = | 1  | 1  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  |
| $h[1]x[n-1]$             | = | 0  | 2  | 2  | 2  | 2  | 0  | 0  | 0  | 0  | 0  |
| $h[2]x[n-2]$             | = | 0  | 0  | 3  | 3  | 3  | 3  | 0  | 0  | 0  | 0  |
| $(x * h)[n]$             | = | 1  | 3  | 6  | 6  | 5  | 3  | 0  | 0  | 0  | 0  |
| $(x \circledast h)_6[n]$ | = | 1  | 3  | 6  | 6  | 5  | 3  | 1  | 3  | 6  | 6  |
| $(x \circledast h)_5[n]$ | = | 4  | 3  | 6  | 6  | 5  | 4  | 3  | 6  | 6  | 5  |
| $(x \circledast h)_4[n]$ | = | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  |
| $(x \circledast h)_3[n]$ | = | 7  | 8  | 9  | 7  | 8  | 9  | 7  | 8  | 9  | 7  |
| $(x \circledast h)_2[n]$ | = | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |
| $(x \circledast h)_1[n]$ | = | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 |

Here's the take-home message for today: You need to get comfortable with performing convolution from this moment on. If you can't perform convolution, you can't perform circular convolution. We will perform convolution for the rest of the class. There will be many convolution problems in the homework and on the quizzes. Going forward, you cannot succeed in this class if you cannot perform convolution.

On the bright side, once you can perform convolution, circular convolution is easy!

## Convolution: Three Ways

The signal  $x[n]$ , defined below, is zero outside the indicated range.



Consider three ways to calculate the convolution of  $x[n]$  with itself.

1. direct convolution:

$$y_1[n] = (x * x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

2. using DTFTs:

$$y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

3. using DFTs of length  $N=16$ :

$$y_3[n] = 16 \sum_{k=0}^{15} X^2[k] e^{j\frac{2\pi k}{16} n} \quad \text{where} \quad X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n] e^{-j\frac{2\pi k}{16} n}$$

## Convolution: Three Ways

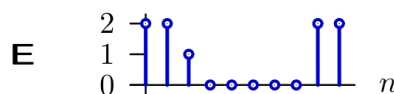
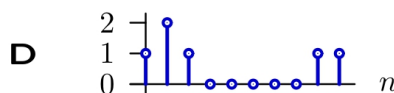
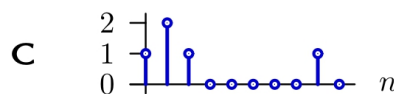
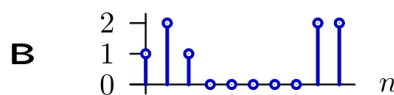
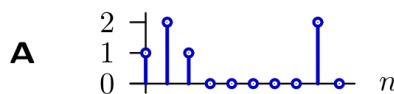
The plots on the right show the **first ten samples** of five signals.

Match the signals on the left with the corresponding plots on the right.

$$y_1 = (x * x) \quad \boxed{\phantom{000}}$$

$$y_2 = \text{DTFT}^{-1}(X^2(\Omega)) \quad \boxed{\phantom{000}}$$

$$y_3 = N \times \text{DFT}^{-1}(X^2[k]) \quad \boxed{\phantom{000}}$$



## Circular Convolution

Multiplication of DFTs corresponds to **circular** convolution in time. Assume that  $F[k]$  is the product of the DFTs of  $f_a[n]$  and  $f_b[n]$ .

$$\begin{aligned}
 f[n] &= \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N}n} \\
 &= \sum_{k=0}^{N-1} F_a[k] \left( \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n} \\
 &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \\
 &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m]
 \end{aligned}$$

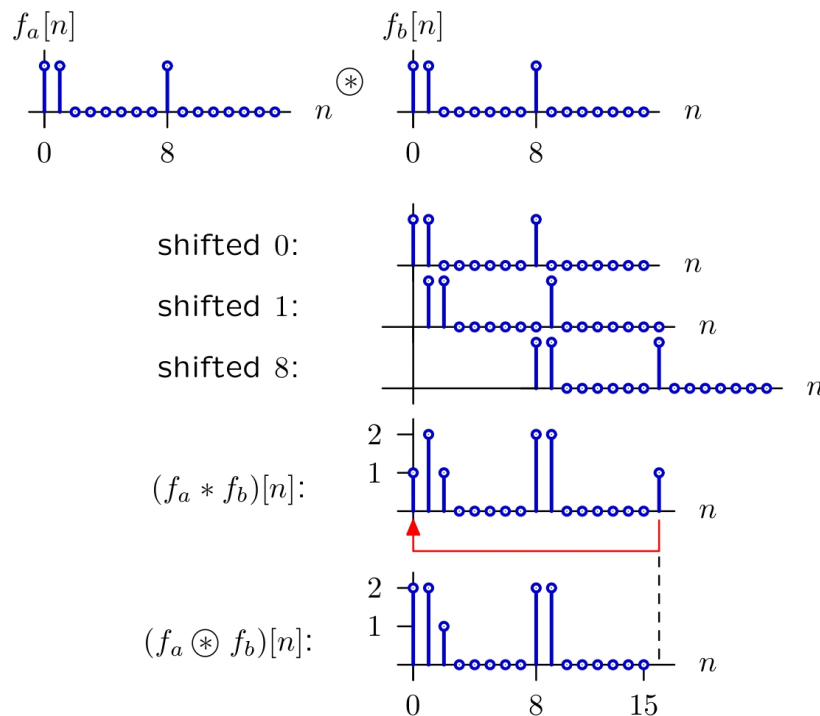
where  $f_{ap}[n] = f_a[n \bmod N]$  is a periodically extended version of  $f_a[n]$ .

We refer to this as **circular** or **periodic** convolution:

$$\frac{1}{N} (f_a \circledast f_b)[n] \xrightarrow{\text{DFT}} F_a[k] \times F_b[k]$$

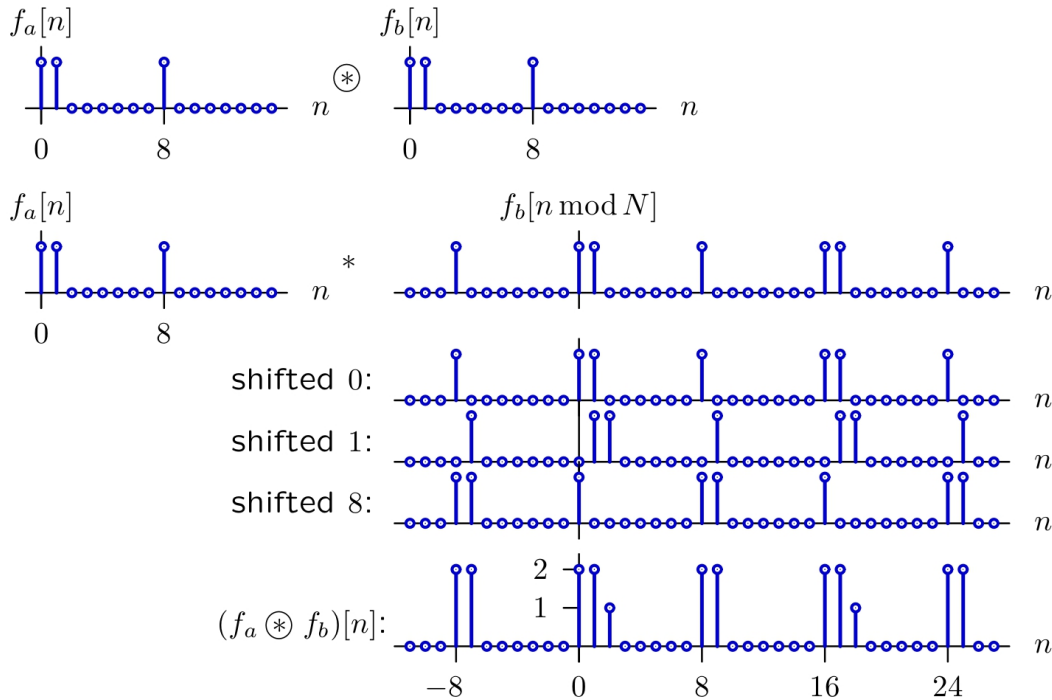
## Circular Convolution

Circular convolution is equivalent to conventional convolution followed by periodic summation of results back into base period.



## Circular Convolution

Circular convolution of two signals is equal to conventional convolution of one signal with a periodically extended version of the other.



## Summary

One of the most useful properties of the DTFT is its filter property: convolution in time corresponds to multiplication in frequency.

$$(f * g)[n] \xrightarrow{\text{DTFT}} F(\Omega)G(\Omega)$$

The DFT is slightly more complicated since the DFT is equivalent to the DTFS of a periodically extended version of  $x[n]$ :

$$x[n] = x[n + mN] \quad \text{for all integers } m$$

A result of this periodicity is that the convolution that results when two DFTs are multiplied is also periodic.

We refer to this type of convolution as “circular convolution.”

$$\frac{1}{N}(f \circledast g)[n] \xrightarrow{\text{DFT}} F[k]G[k]$$