6.300: Signal Processing

Tutorial Topic Discrete Fourier Transform
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Discrete Fourier Transform

We may represent a periodic discrete-time signal $x[n] = x[n+N]$ as a finite-length sum of harmonically-related complex exponentials with a discrete-time Fourier series. Periodicity is rather restrictive, as many real-world signals of interest are not periodic. To be periodic, a signal must repeat forever!

The discrete-time Fourier transform is a Fourier representation for aperiodic discrete-time signals $x[n]$, but the analysis formula entails an infinite-length sum, and the spectrum $X(\Omega)$ — a function of the continuous variable Ω — is continuous. While a useful theoretical tool, the DTFT is also somewhat impractical.

In search of a Fourier representation that is amenable for digital computation, we turn to the DFT. You might call the DFT a discrete-time, discrete-frequency Fourier transform.

- finite-length signals — windowing $x_w[n] = x[n]w[n]$
- discrete in time — time indexed by integer n
- discrete in frequency — frequency indexed by integer k

Analysis: $$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$$

Synthesis: $$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}$$

Note: There is no general consensus on where to place the $1/N$ factor in the equations that define the DFT. In 6.300, we place the $1/N$ factor in the analysis equation, so that the DFT and DTFS analysis equations match. In both cases, then, the DC term $X[0]$ is the average value of $x[n]$ over a length-N interval. Some authors and numerical packages (e.g., MATLAB) place the $1/N$ factor in the synthesis equation instead, however. Other authors and numerical packages even put a factor of $1/\sqrt{N}$ in both the analysis and synthesis equations to make the DFT a unitary transformation.

Relation to Discrete-Time Fourier Series (DTFS)

The length-N DFT is equivalent to the DTFS of an N-periodic extension of the windowed signal $x_w[n] = x[n]w[n]$. From this perspective, sharp discontinuities in the periodic extension of $x_w[n]$ lead to spurious high-frequency content spread across the spectrum of $x_w[n]$.

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x_w[(n \bmod N)]e^{-j2\pi kn/N}$$

Relation to Discrete-Time Fourier Transform (DTFT)

The DFT returns N scaled, equally-spaced samples of the DTFT over the interval $[0, 2\pi)$. Increasing the DFT length N yields more samples of the DTFT, which are spaced more closely together.

$$X[k] = \tfrac{1}{N}X_w\left(\tfrac{2\pi}{N}k\right)$$

cyclical frequency resolution: $\dfrac{f_s}{N}$ hertz   angular frequency resolution: $\dfrac{2\pi}{N}$ radians

Single Sinusoid

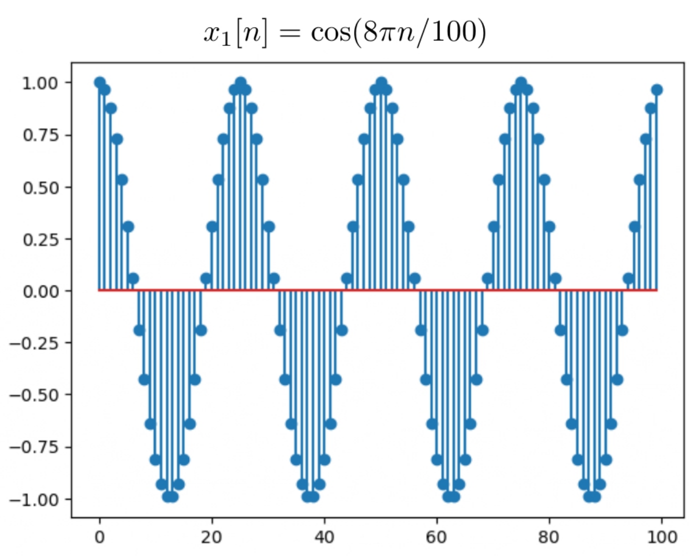
Create four signals

$$x_1[n] = \cos(8\pi n/100)$$
$$x_2[n] = \cos(8\pi n/100 - \pi/4)$$
$$x_3[n] = \cos(9\pi n/100)$$
$$x_4[n] = \cos(9\pi n/100 - \pi/2)$$

each with a duration of 1 second when the sample frequency is $44,100$ Hz.

Compare the DFTs of the first 100 samples of each of these signals.

Single Sinusoid



What is the frequency of this tone if the sample rate is $44,100$ Hz?

Single Sinusoid

What is the frequency of the tone generated by $x_1[n]$?

Since $x_1[n] = \cos(8\pi n/100)$, we know that the discrete frequency $\Omega_1 = \frac{8\pi}{100}$. Furthermore, the sample frequency $f = f_s$ corresponds to the maximum discrete frequency $\Omega = 2\pi$, and frequencies f in Hz are proportional to discrete frequencies Ω.

$$\frac{f}{f_s} = \frac{\Omega}{2\pi}$$

So $f = \frac{\Omega f_s}{2\pi} = \frac{8\pi/100}{2\pi} \times 44,100\,\text{Hz} = 1764\,\text{Hz}$

				frequency
0	1	$N-1$	N	k
0	$\frac{2\pi}{N}$		2π	Ω [rad/sample]
0	$\frac{f_S}{N}$		f_S	f [Hz]

Single Sinusoid

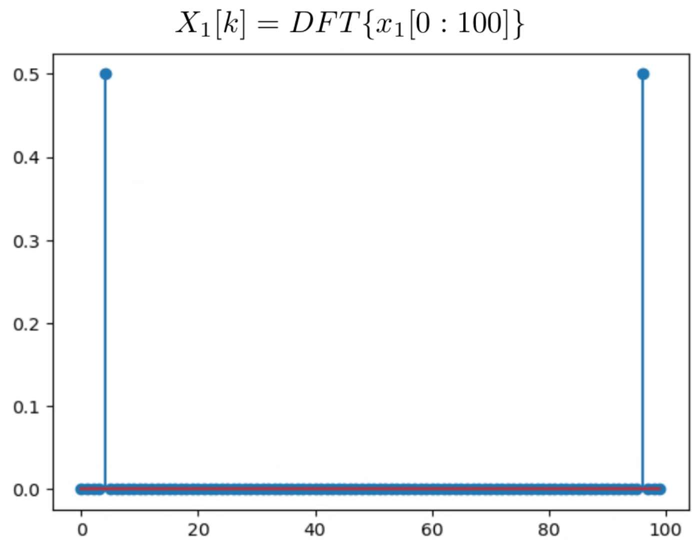
Write a program to calculate the DFT of an input sequence.
Use that program to calculate $X_1[k]$, which is the DFT of the first 100 samples of $x_1[n]$.

```
def dft(x):
    N = len(x)
    answer = [0 for k in range(N)]
    for k in range(N):
        for n in range(N):
            answer[k] += (1/N)*x[n]*e**(-2j*pi*k*n/N)
    return answer

X1 = dft(x1[0:100])
```

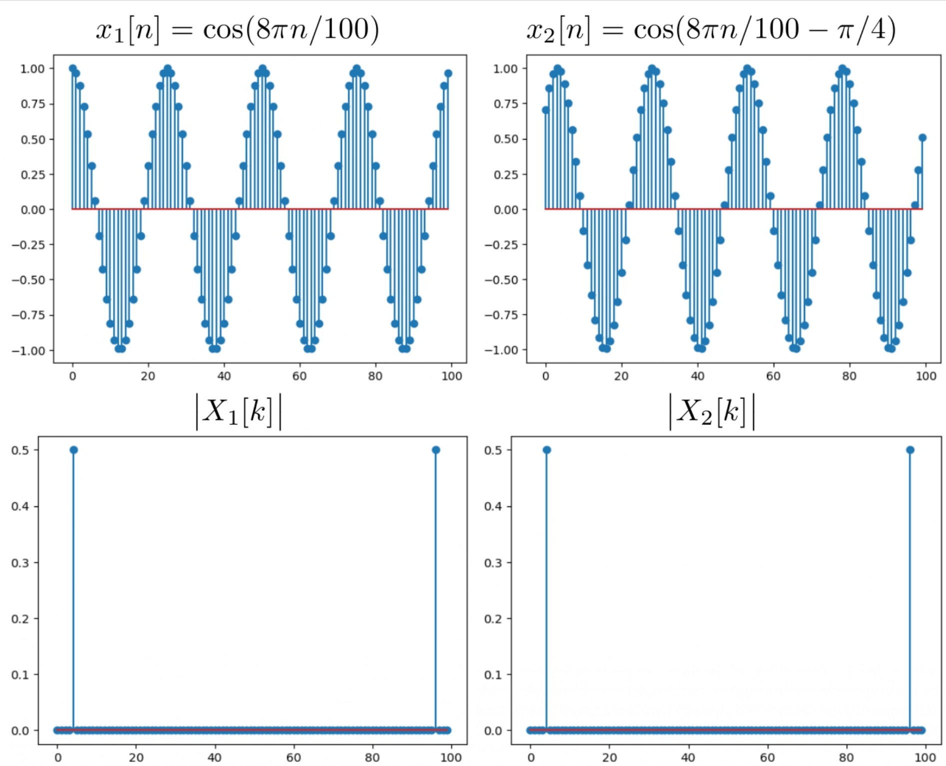
Single Sinusoid

Plot the magnitude of $X_1[\cdot]$.



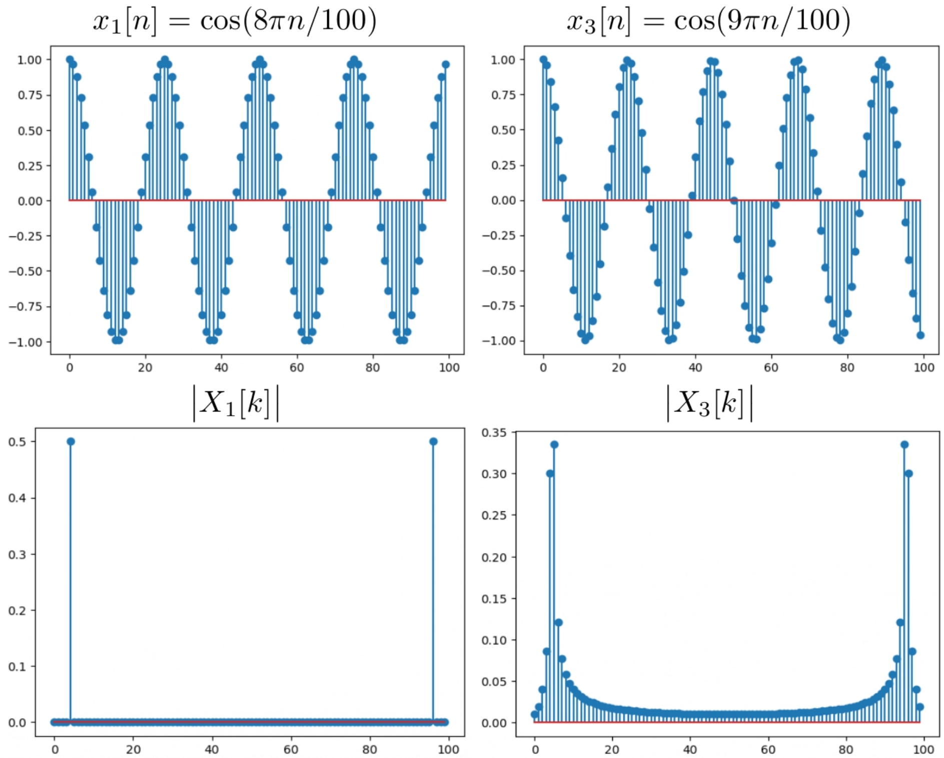
Which values of k are non-zero?

Compare Two Signals



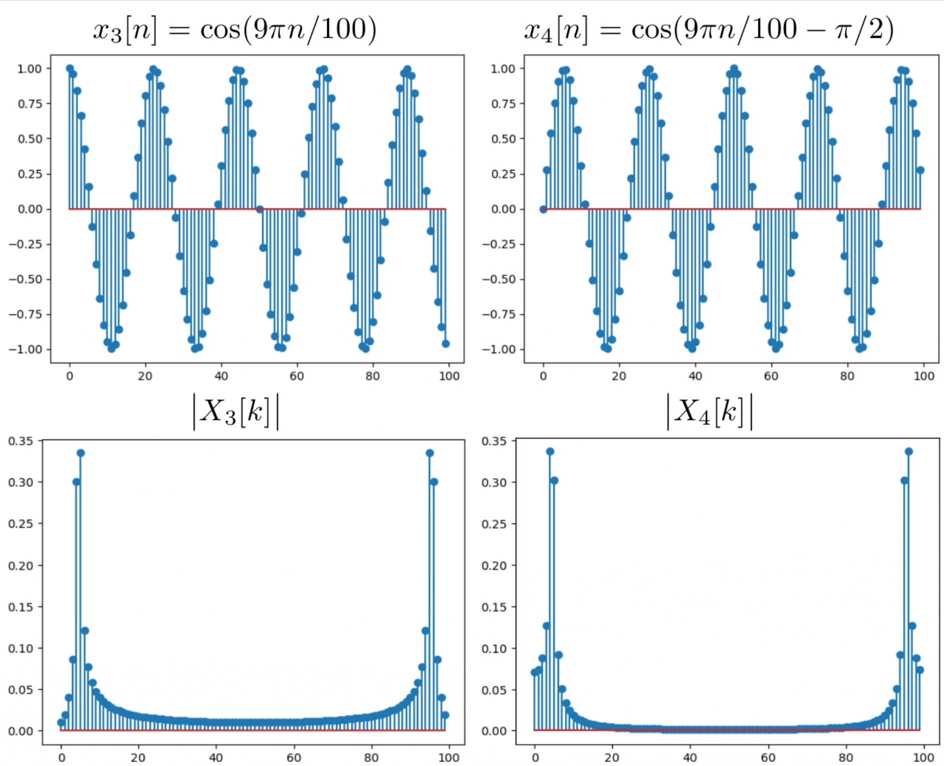
No difference in magnitudes (but the phases are different).

Compare Two Signals



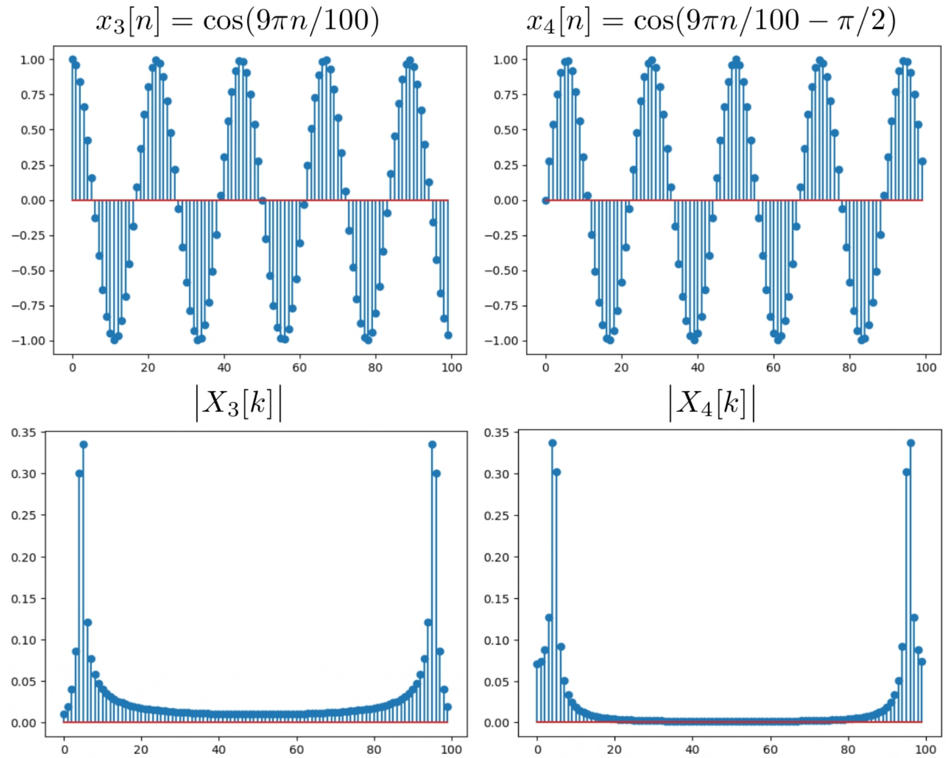
Why are these DFTs so different?

Compare Two Signals



$\Omega_3 = \Omega_4$. But DC bigger: 5 positive half cycles versus 4 negative ones.

Compare Two Signals

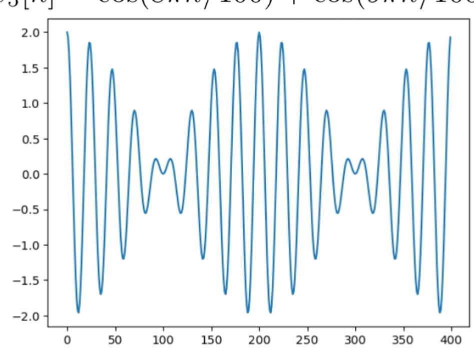


High freq. content of X_4 smaller than X_3: $|x_4[99] - x_4[0]| < |x_3[99] - x_3[0]|$

Analyzing Signals with Multiple Frequencies

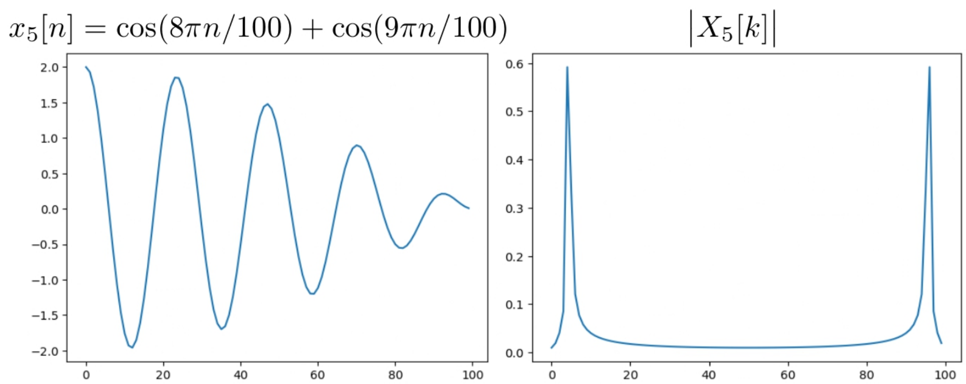
What is the minimum window size N needed to resolve $\Omega = 8\pi/100$ from $9\pi/100$?

$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$



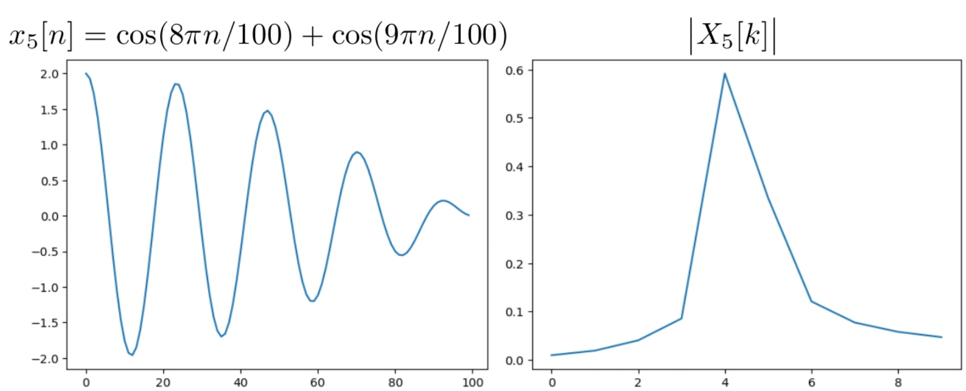
Analyzing Signals with Multiple Frequencies

If the analysis window is small (here $N{=}100$), the two frequencies $8\pi/100$ and $9\pi/100$ generate a single peak in the DFT at $k = 4$ (along with its partner at $k = 100{-}4 = 96$).



Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.
$N = 100$ zoomed

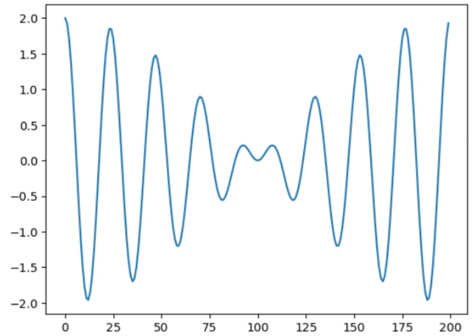


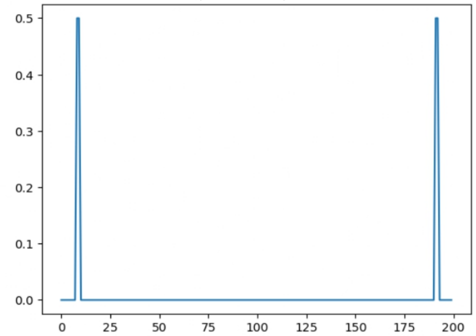
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

$N = 200$

$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$ $\qquad$ $\big|X_5[k]\big|$



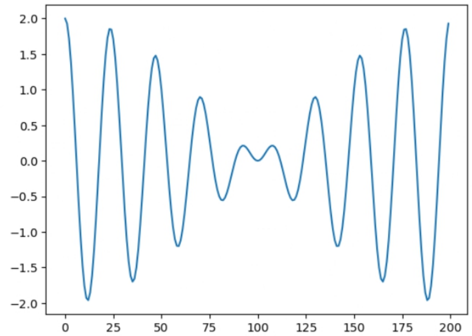


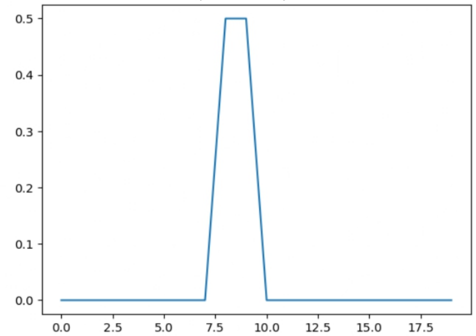
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

$N = 200$ zoomed

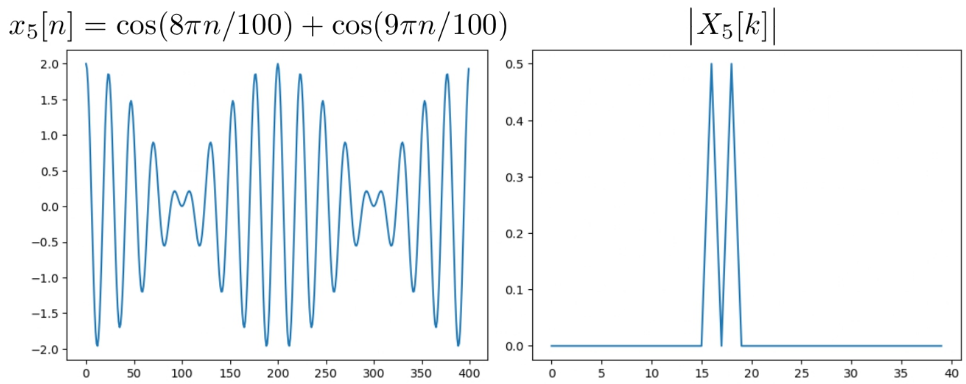
$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$ $\qquad$ $\big|X_5[k]\big|$





Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.
$N = 400$ zoomed

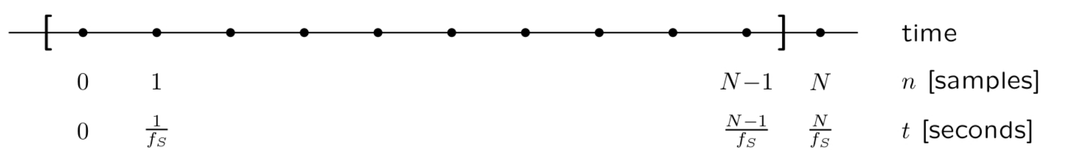


These frequencies are clearly resolved with $N = 400$.

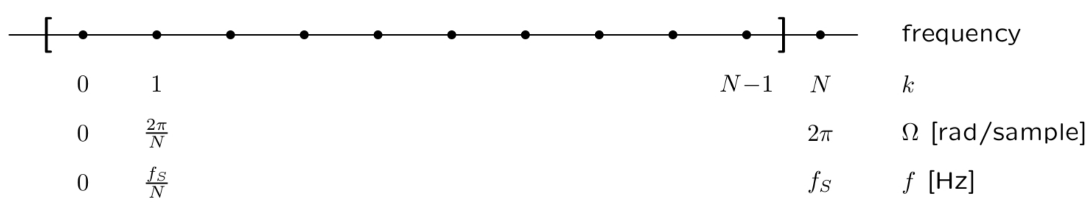
Frequency Scales

We can think of the DFT as having spectral resolution of $(2\pi/N)$ radians, which is equivalent to (f_S/N) Hz.

The time window is divided into N samples numbered $n = 0$ to $N-1$.



Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.



Analyzing Signals with Multiple Frequencies

Two frequencies are resolved if they are separated by more than $\frac{2\pi}{N}$.

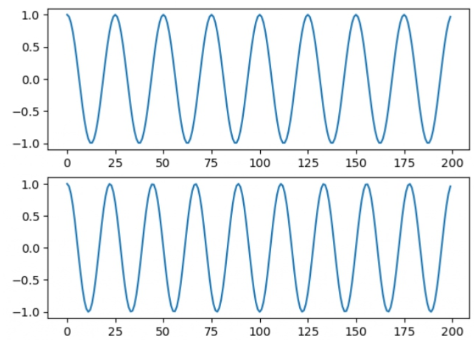
$\Omega_1 = \frac{8\pi}{100}$ and $\Omega_2 = \frac{9\pi}{100}$ will be resolved if

$$\Delta\Omega = \Omega_2 - \Omega_1 = \frac{9\pi}{100} - \frac{8\pi}{100} = \frac{\pi}{100} > \frac{2\pi}{N}$$

That is, if $N > 200$.

We can think of $\frac{2\pi}{N}$ as the frequency resolution of the DFT.

Notice 8 full cycles of Ω_1 and 9 full cycles of Ω_2 fit in $N = 200$.



Summary

Time and frequency resolution are important issues in all Fourier analyses.

Frequency resolution is determined by the number of samples N included in the analysis.