6.300: Signal Processing

Tutorial Topics Frequency Response and Filtering
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Frequency Response

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega t} \to \boxed{\text{LTI}} \to H(\omega)e^{j\omega t} = |H(\omega)|e^{j(\omega t + \angle H(\omega))} \qquad X(\omega) \to \boxed{\text{LTI}} \to Y(\omega) = H(\omega)X(\omega)$$

$$e^{j\Omega n} \to \boxed{\text{LTI}} \to H(\Omega)e^{j\Omega n} = |H(\Omega)|e^{j(\Omega n + \angle H(\Omega))} \qquad X(\Omega) \to \boxed{\text{LTI}} \to Y(\Omega) = H(\Omega)X(\Omega)$$

We may characterize an LTI system as a *filter* that shapes a signal's spectrum.

Here is the physical interpretation: If the input to an LTI system is a sinusoid, the output of the LTI system is a scaled (i.e., amplified or attenuated) and phase-shifted sinusoid.

$$\cos(\omega t) \to \boxed{\text{LTI}} \to |H(\omega)| \cos(\omega t + \angle H(\omega))$$

$$\cos(\Omega n) \to \boxed{\text{LTI}} \to |H(\Omega)| \cos(\Omega n + \angle H(\Omega))$$

Convolution Theorem

Convolution in time corresponds to multiplication in frequency.
Convolution in frequency corresponds to multiplication in time.

$$(x * h)(t) \longleftrightarrow X(\omega)H(\omega) \qquad x(t)h(t) \longleftrightarrow \tfrac{1}{2\pi}(X * H)(\omega)$$

$$(x * h)[n] \longleftrightarrow X(\Omega)H(\Omega) \qquad x[n]h[n] \longleftrightarrow \tfrac{1}{2\pi}(X * H)(\Omega)$$

Difference Equation vs. Impulse Response vs. Frequency Response

Consider the LTI system represented by the following difference equation.

$$\sum_{k=-\infty}^{\infty} a_k y[n-k] = \sum_{k=-\infty}^{\infty} b_k x[n-k]$$

Computing the DTFT yields the frequency response.

$$\sum_{k=-\infty}^{\infty} a_k e^{-jk\Omega} Y(\Omega) = \sum_{k=-\infty}^{\infty} b_k e^{-jk\Omega} X(\Omega) \longleftrightarrow H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=-\infty}^{\infty} b_k e^{-jk\Omega}}{\sum_{k=-\infty}^{\infty} a_k e^{-jk\Omega}}$$

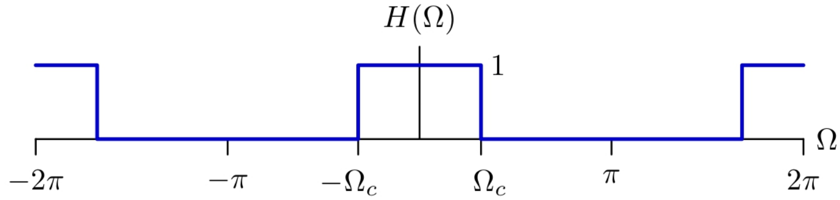
The impulse response is given by the inverse DTFT of the frequency response. $H(\Omega)$ is a rational polynomial that one can decompose into a sum of simpler rational polynomials (e.g., via partial fractions) with simpler inverse DTFTs.

$$h[n] = \frac{1}{2\pi} \int_{2\pi} H(\Omega) e^{j\Omega n} d\Omega$$

$$\text{e.g., } h[n] = \frac{1}{2\pi} \int_{2\pi} \left(\frac{1}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 + \frac{1}{3}e^{-j\Omega}} \right) e^{j\Omega n} d\Omega = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]$$

The "Ideal" Low-Pass Filter

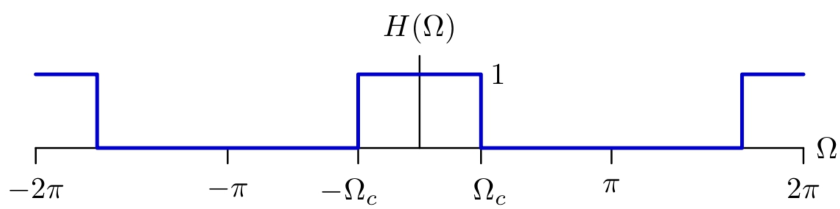
Consider a system characterized by the following purely real frequency response:



Such a system is called a **low-pass filter**, because it allows low frequencies to pass through unmodified, while attenuating high frequencies.

We could apply this filter to a signal by multiplying the DTFT of that signal by the values above. But we could also apply the filter by operating in the time domain.

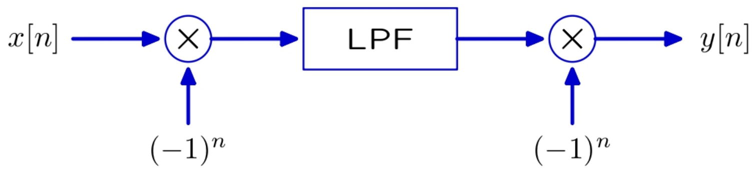
The "Ideal" Low-Pass Filter



We can apply this filter to a signal by convolving with its unit sample response. What is the unit sample response of the system whose frequency response is shown above?

Cascaded System

Consider the following system, where LPF represents a lowpass filter of the form discussed on the previous slides.



How many of the following statements are true?

- The transformation from $x[n]$ to $y[n]$ is linear.
- The transformation from $x[n]$ to $y[n]$ is time invariant.
- The transformation from $x[n]$ to $y[n]$ is a high-pass filter.

Hint #1: Analyze each system component separately.

Is the multiplier $g[n] = (-1)^n f[n]$ linear and time-invariant?

Is the low-pass filter $f[n] \rightarrow \boxed{\textbf{LPF}} \rightarrow g[n] = (f * h_{\text{LPF}})[n]$ linear and time-invariant?

If all three system components are linear, the composite system is linear.
If any system components are nonlinear, can the composite system be linear?

If all three system components are time-invariant, the composite system is time-invariant.
If any system components are time-variant, can the composite system be time-invariant?

Hint #2: Determine the unit-sample response $h_{\text{LPF}}[n]$ corresponding to frequency response

$$H_{\text{LPF}}(\Omega) = H_{\text{LPF}}(\Omega + 2\pi) = \begin{cases} 1 & -\Omega_c \leq \Omega \bmod 2\pi \leq \Omega_c \\ 0 & \text{otherwise.} \end{cases}$$

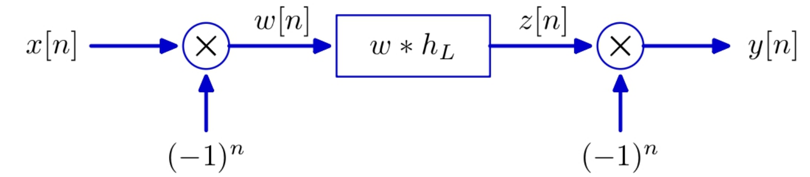
Then determine the frequency response $H_{\text{HPF}}(\Omega)$ corresponding to the unit-sample response

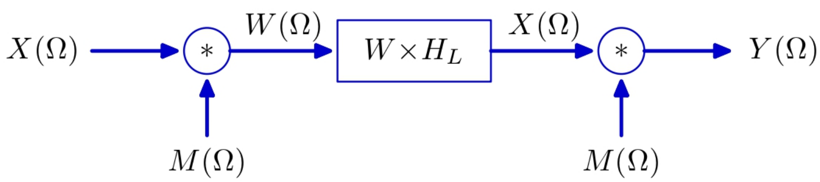
$$h_{\text{HPF}}[n] = (-1)^n h_{\text{LPF}}[n].$$

Sketch $H_{\text{LPF}}(\Omega)$ and $H_{\text{HPF}}(\Omega)$ over $-2\pi \leq \Omega \leq 2\pi$.

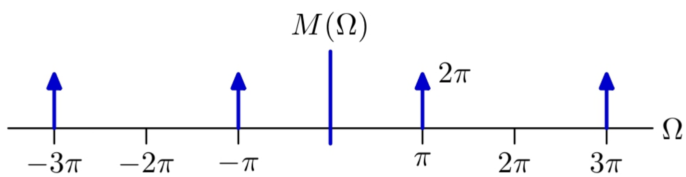
Cascaded System

Alternatively, we could solve this problem in the frequency domain.





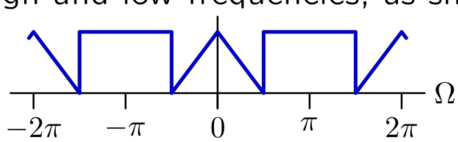
Since $(-1)^n = e^{j\pi n}$, $M(\Omega)$ is a complex sinusoid with frequency π.



Cascaded System

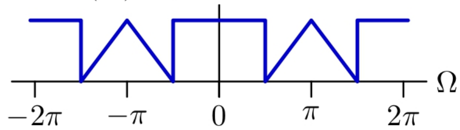
Assume $X(\Omega)$ differs at high and low frequencies, as shown below.

$X(\Omega)$:



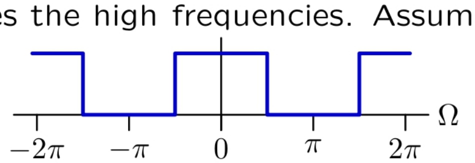
Convolving with $M(\Omega)$ shifts $X(\Omega)$ by $\Omega = \pi$:

$W(\Omega)$:



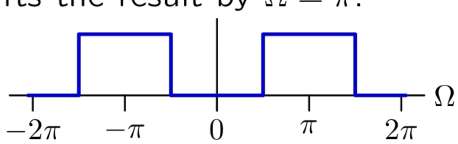
Low pass filtering removes the high frequencies. Assume $\Omega_c = \pi/2$:

$Z(\Omega)$:



Convolving with $M(\Omega)$ shifts the result by $\Omega = \pi$:

$Y(\Omega)$:



$Y(\Omega)$ is a highpass version of $X(\Omega)$.