6.300: Signal Processing

Tutorial Topics Convolution and Impulse Response

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Convolution

Convolution is a mathematical operation. It is associative, commutative, and distributive.
In discrete time, convolution takes the form of a summation.

$$\textbf{Convolution Sum: } (x*h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = (h*x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Compute the result using superposition. Use a table or sketch a plot to keep track of the values.

n	=	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$x[n]$	=	1	1	1	1	0	0	0	0	0	0
$h[n]$	=	1	2	3	0	0	0	0	0	0	0

n	=	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$h[0]\,x[n-0]$	=	1	1	1	1	0	0	0	0	0	0
$h[1]\,x[n-1]$	=	0	2	2	2	2	0	0	0	0	0
$h[2]\,x[n-2]$	=	0	0	3	3	3	3	0	0	0	0
$(x*h)[n]$	=	**1**	**3**	**6**	**6**	**5**	**3**	**0**	**0**	**0**	**0**

In continuous time, convolution takes the form of an integral.

$$\textbf{Convolution Integral: } (x*h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = (h*x)(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

It's probably simplest to evaluate the integral piecewise over several intervals.

Unit-Sample Response

$$\text{unit-sample signal } \delta[n] \rightarrow \boxed{\text{DT LTI System}} \rightarrow \text{unit-sample response } h[n] = \sum_{k=-\infty}^{\infty} h[k]\delta[n-k]$$

$$\text{e.g., } y[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-1] \text{ with initial rest conditions } \iff h[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1]$$

$$\text{e.g., } y[n] = \tfrac{1}{2}y[n-1] + x[n] \text{ with initial rest conditions } \iff h[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

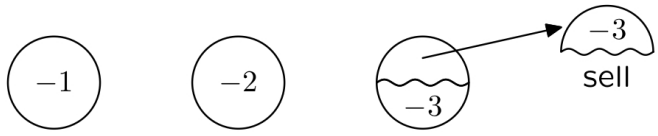
Impulse Response

$$\text{unit-impulse signal } \delta(t) \rightarrow \boxed{\text{CT LTI System}} \rightarrow \text{impulse response } h(t) = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau$$

Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

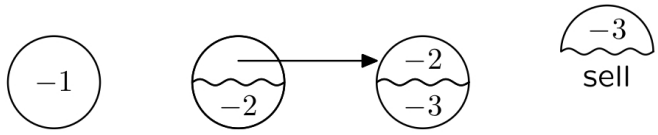
Sell half of the oldest stock.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

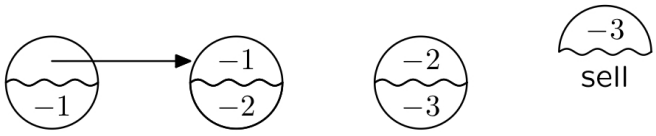
Refill oldest barrel from next-to-oldest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

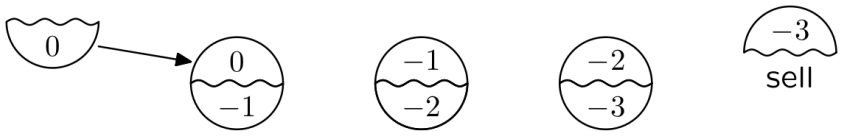
Refill next-to-oldest barrel from youngest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

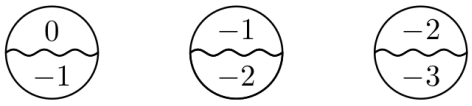
Refill youngest barrel with this year's harvest.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.

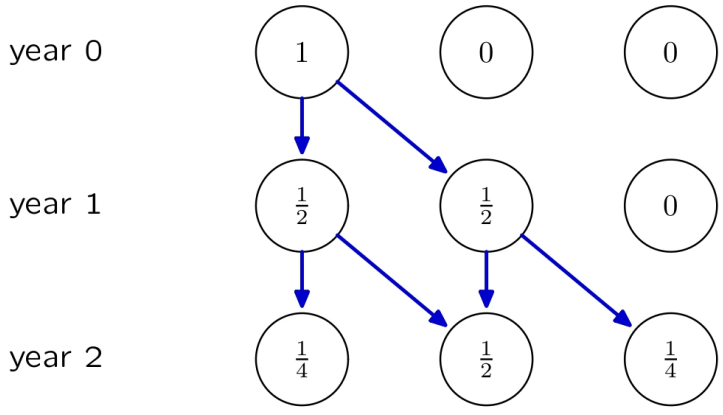


Properties of solera process:

- Mixing produces a more uniform product.
- Mitigates worst-case results of one bad year.
- Blends wines from MANY previous years.

Solera Analysis

We can analyze these effects with a tracer experiment.
Add 1 unit of tracer to new crop; track tracer through the system.

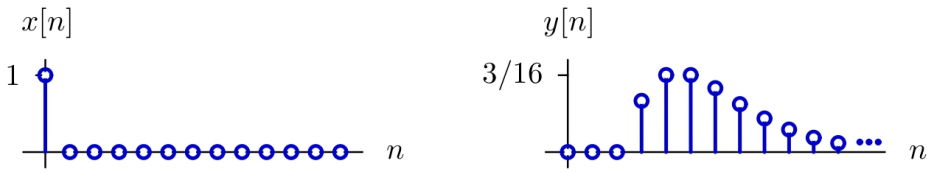


How much tracer will be in each barrel at the end of year 3?

Solera Analysis

Add 1 unit of tracer to new crop; track tracer through the system.

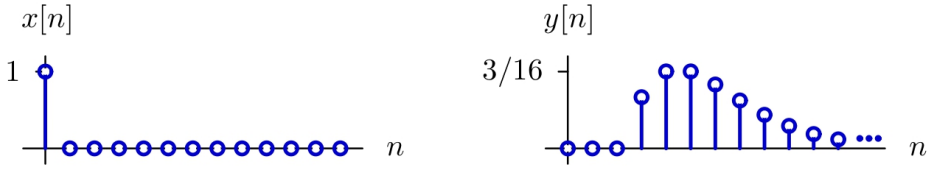
Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64



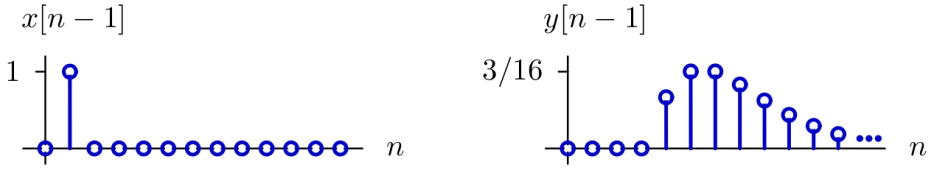
Solera Analysis

How would results change if tracer were added in year 1 (not 0)?

Original response:



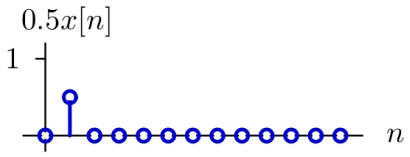
Delayed input $\rightarrow$ delayed output:

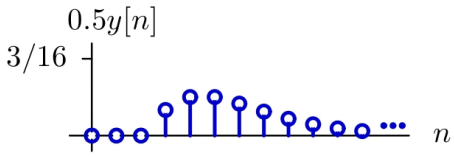


Delaying the input by a year simply delays the outputs by one year.

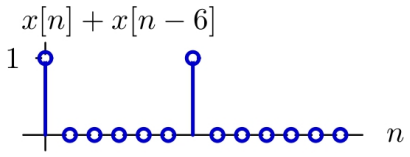
Solera Analysis

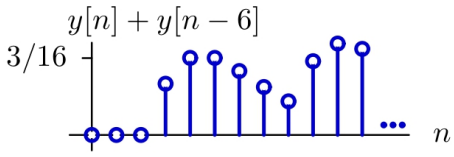
Scaling the input amplitudes:



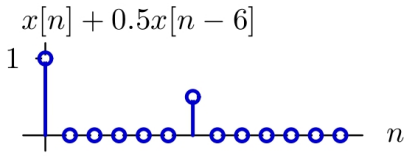


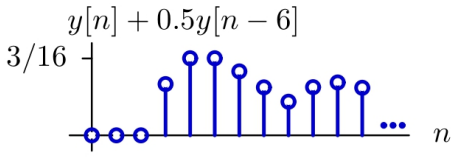
Adding two inputs:





Linearly combining two inputs:





Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2?

1. $\frac{21}{32}$ 2. $\frac{1}{2}$ 3. $\frac{3}{16}$ 4. $\frac{9}{16}$ 5. none of above