6.300: Signal Processing

Tutorial Topic Continuous-Time Fourier Transform
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Continuous-Time Fourier Transform (CTFT)

The continuous-time Fourier transform may be conceptualized as the continuum limit of a continuous-time Fourier series. Infinitely-many discrete harmonics $k\omega_0$ cluster infinitely-close together to form a continuous frequency spectrum: $k\omega_0$ (function of integer k) $\mapsto \omega$ (function of real-valued ω).

Synthesis: $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ **Analysis:** $X(\omega) = \displaystyle\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Fourier Transform Pairs

$$\delta(t) \iff 1 \text{ (for all } \omega) \qquad 1 \text{ (for all } t) \iff 2\pi\delta(\omega)$$

$$\delta(t - t_0) \iff e^{-j\omega t_0} \qquad e^{j\omega_0 t} \iff 2\pi\delta(\omega - \omega_0)$$

Fourier Transform Properties

You fill this in! Many properties that we've seen in the context of Fourier series still hold true.
After you solve the problems on the other side of this sheet, write down the properties you just derived.

Fourier Series vs. Fourier Transform for Periodic Signals

Series: $f(t) \iff F[k]$ **Transform:** $f(t) \iff \displaystyle\sum_k \underbrace{2\pi F[k]\delta(\omega - k\omega_0)}_{\text{impulses at harmonics}}$

Continuous-Time Fourier Transform

(a) Determine $X_1(\omega)$, the continuous-time Fourier transform (CTFT) of $x_1(t)$.

$$x_1(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Sketch the real part $\text{Re}\{X_1(\omega)\}$ and imaginary part $\text{Im}\{X_1(\omega)\}$ of $X_1(\omega)$ on separate plots.

Sketch the magnitude $|X_1(\omega)|$ and phase $\angle X_1(\omega)$ on separate plots.

(b) Determine $X_2(\omega)$, the Fourier transform of $x_2(t)$.

$$x_2(t) = x_1(t - t_0)$$

Sketch the magnitude $|X_2(\omega)|$ and phase $\angle X_2(\omega)$ on separate plots. **Hint:** You ought to be able to re-use much of the work you did in **(a)** if you apply your knowledge of Fourier transform properties.

(c) Determine $X_3(\omega)$, the Fourier transform of $x_3(t)$.

$$x_3(t) = \text{Symmetric}\{x_1(t)\}$$

A time-domain signal that is real and symmetric has a Fourier transform that is _____ and _____.

(d) Determine $X_4(\omega)$, the Fourier transform of $x_4(t)$.

$$x_4(t) = \text{Antisymmetric}\{x_1(t)\}$$

A time-domain signal that is real and antisymmetric has a Fourier transform that is _____ and _____.

(e) Determine $X_5(\omega)$, the Fourier transform of $x_5(t)$.

$$x_5(t) = \frac{d}{dt}\Big[\text{Symmetric}\{x_1(t)\}\Big] = \frac{dx_3(t)}{dt}$$

Hint: Start by writing out the CTFT synthesis equation. Differentiate both sides with respect to t.

Inverse Continuous-Time Fourier Transform

(f) Determine $x_6(t)$, the time-domain signal whose Fourier transform is $X_6(\omega)$.

$$X_6(\omega) = e^{-|\omega|}$$

Hint: Directly apply the CTFT synthesis equation. Split the synthesis integral over $(-\infty, \infty)$ into two integrals — one integral over $(-\infty, 0]$ where $|\omega| = -\omega$ and one integral over $[0, \infty)$ where $|\omega| = \omega$.