6.300: Signal Processing

Tutorial Topic Discrete-Time Fourier Series
Author Titus K. Roesler (tkr@mit.edu)

Fourier Series Formulæ

Continuous-Time Fourier Series (CTFS)

$f(t)$ is a T-periodic function with fundamental angular frequency $\omega_0 = 2\pi/T$.

$$\textbf{Synthesis: } f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \textbf{Analysis: } a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}$$

Discrete-Time Fourier Series (DTFS)

$f[n]$ is an N-periodic sequence with fundamental angular frequency $\Omega_0 = 2\pi/N$.

$$\textbf{Synthesis: } f[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \qquad \textbf{Analysis: } a_k = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_0 n}$$

A Few Properties of Fourier Series

Here are a few properties of Fourier series that we'll use often in this class. We'll learn more over time.

Linearity

$$f_1(t) \iff F_1[k] \qquad f_2(t) \iff F_2[k] \qquad f(t) = \alpha f_1(t) + \beta f_2(t) \iff F[k] = \alpha F_1[k] + \beta F_2[k]$$

$$f_1[n] \iff F_1[k] \qquad f_2[n] \iff F_2[k] \qquad f[n] = \alpha f_1[n] + \beta f_2[n] \iff F[k] = \alpha F_1[k] + \beta F_2[k]$$

Time Shift

$$f(t) \iff F[k] \qquad f(t-t_0) \iff F[k]e^{-jk\omega_0 t_0} = |F[k]|e^{j(\angle F[k] - k\omega_0 t_0)}$$

$$f[n] \iff F[k] \qquad f[n-n_0] \iff F[k]e^{-jk\Omega_0 n_0} = |F[k]|e^{j(\angle F[k] - k\Omega_0 n_0)}$$

Time Flip

$$f(t) \iff F[k] \qquad f(-t) \iff F[-k]$$

$$f[n] \iff F[k] \qquad f[-n] \iff F[-k]$$

Conjugate Symmetry (Hermitian Symmetry)

Real-valued signals have conjugate-symmetric Fourier series coefficients.

real-valued $f(t) \iff F[k]$ such that $F^*[k] = F[-k]$ where * denotes complex conjugation

real-valued $f[n] \iff F[k]$ such that $F^*[k] = F[-k]$ where * denotes complex conjugation

6.3000: Signal Processing

Discrete-Time Fourier Series

Synthesis Equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k\, e^{j\frac{2\pi k}{N}n}$$

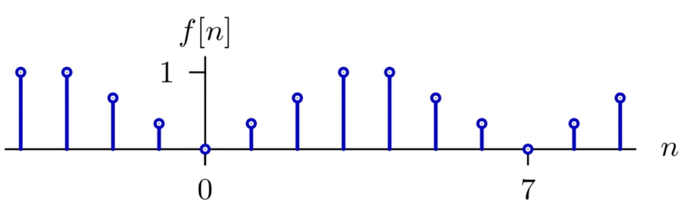
Analysis Equation

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]\, e^{-j\frac{2\pi k}{N}n}$$

February 20, 2025

Find the DT Fourier Series Coefficients

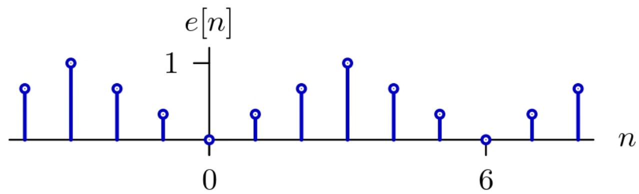
Let $f[n]$ represent a periodic DT signal with period $N = 7$:



Determine the Fourier series coefficients $F[k]$ for $f[n]$.

Find the DT Fourier Series Coefficients

How would the answer change if the period were $N = 6$?

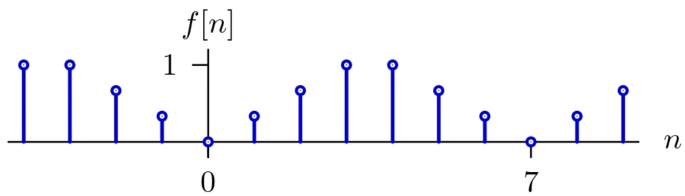


Determine the Fourier series coefficients $E[k]$ for $e[n]$.

Find the DT Fourier Series Coefficients

Consider a new signal $g[n]$ derived from $f[n]$ as follows:

$$g[n] = 9 - 3f[n-1]$$



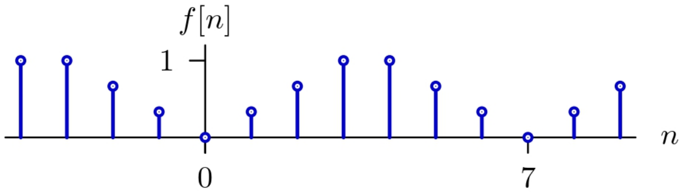
Find the DTFS coefficients of $g[n]$.

Find the DT Fourier Series Coefficients

Consider another new signal

$$h[n] = (-1)^n f[n]$$

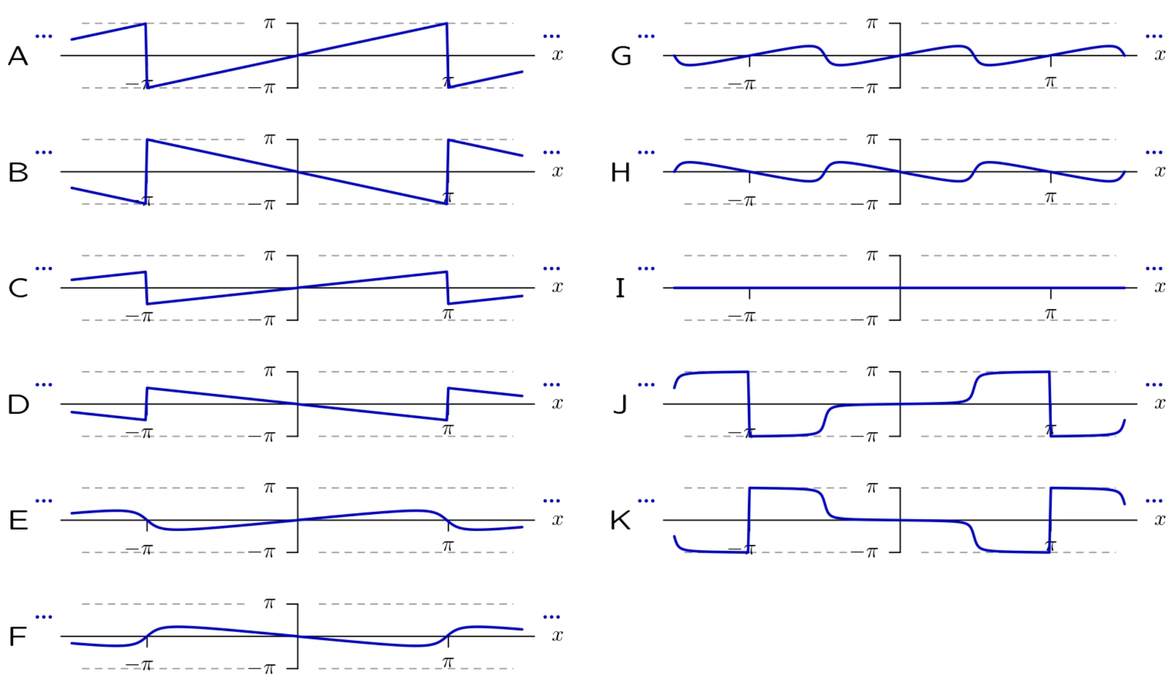
where



Find the DTFS coefficients of $h[n]$.

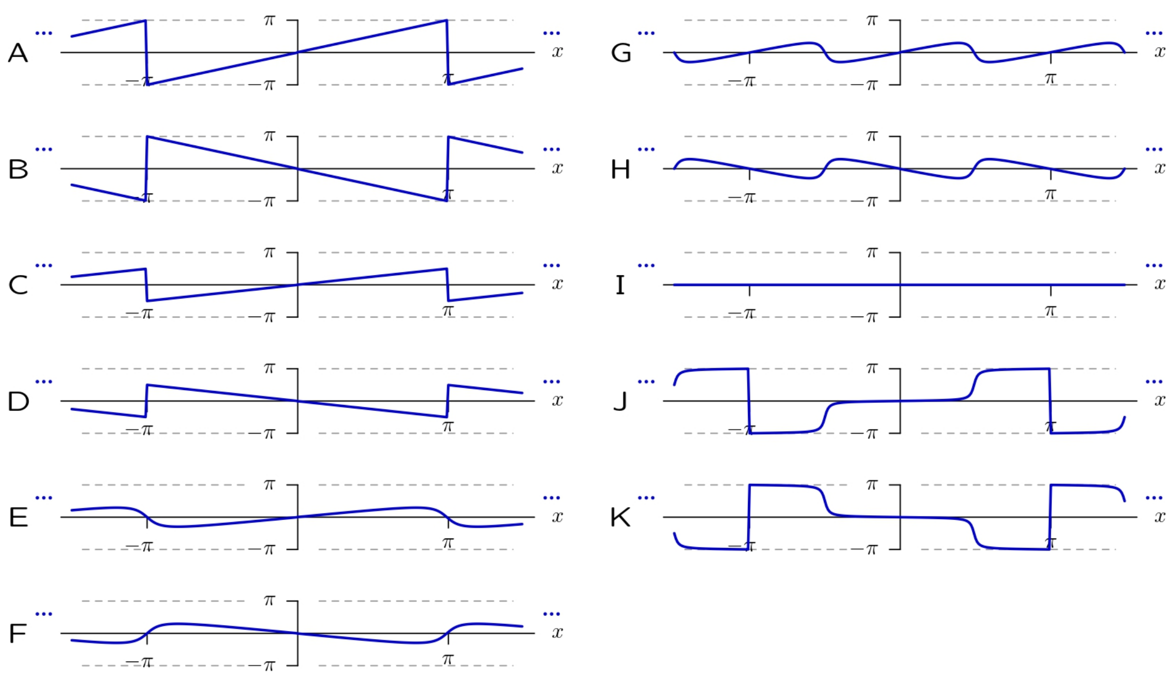
Angular Trends

Which of the following plots shows the angle of e^{-jx}?



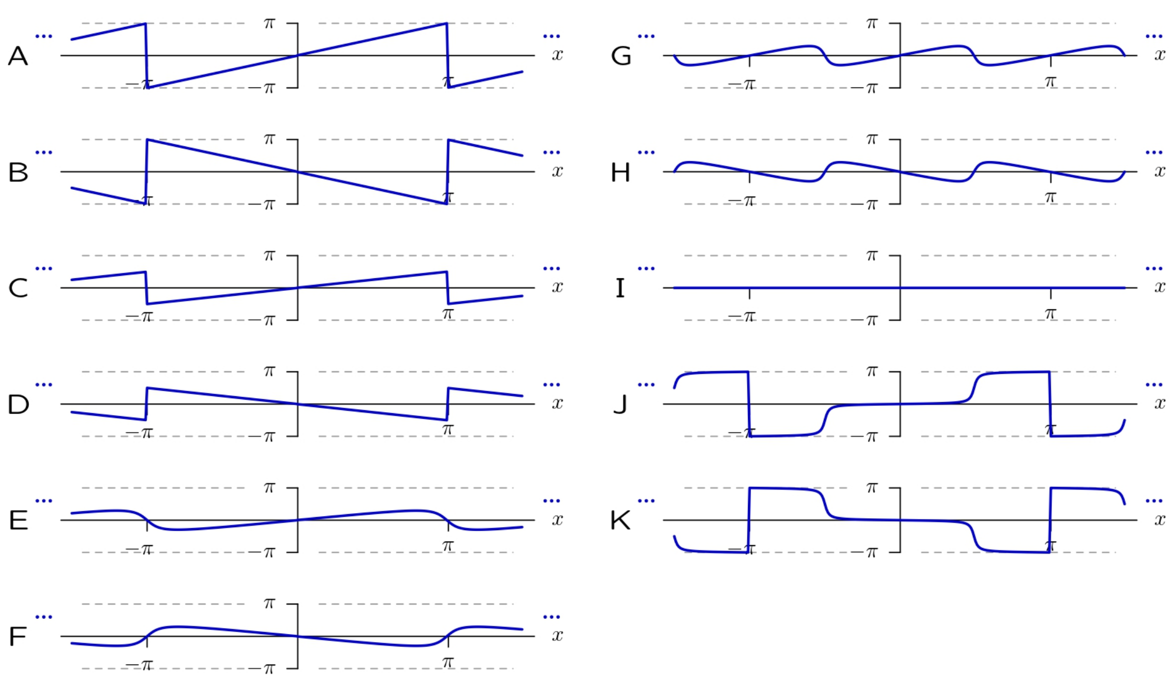
Angular Trends

Which of the following plots shows the angle of $(1 + 0.8e^{jx})$?



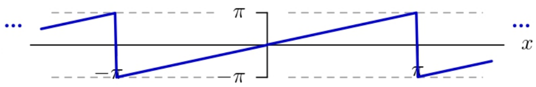
Angular Trends

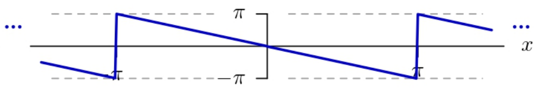
Which of the following plots shows the angle of $\left(\frac{1+0.4e^{jx}}{2+0.8e^{jx}}\right)$?

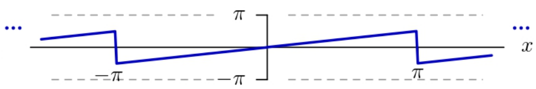


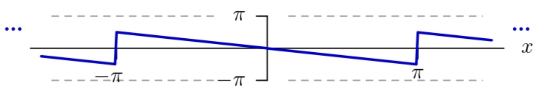
Angular Trends

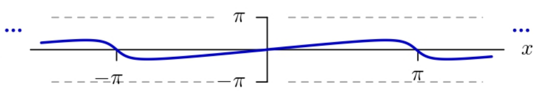
Which of the following plots shows the angle of $(1 + e^{jx})$?

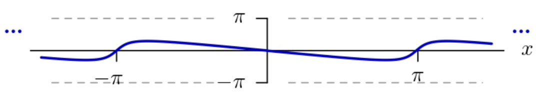
A 

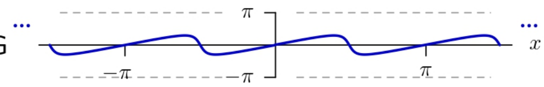
B 

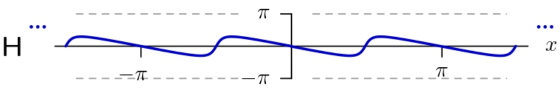
C 

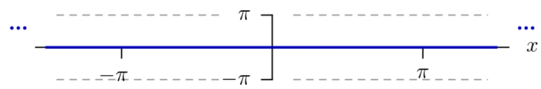
D 

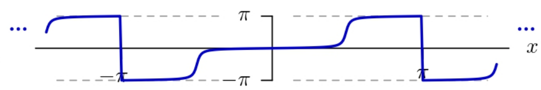
E 

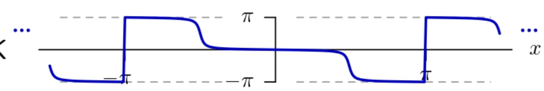
F 

G 

H 

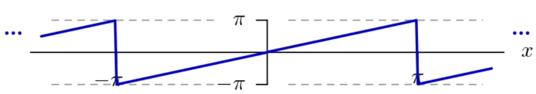
I 

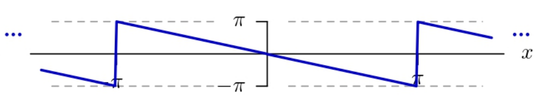
J 

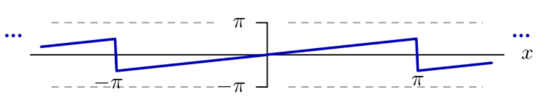
K 

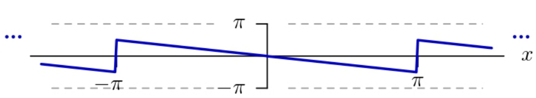
Angular Trends

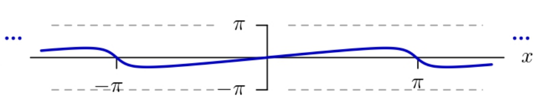
Which of the following plots shows the angle of $(1 + 0.8e^{j2x})$?

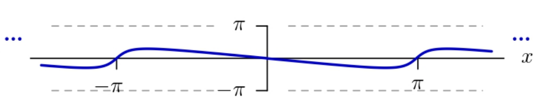
A 

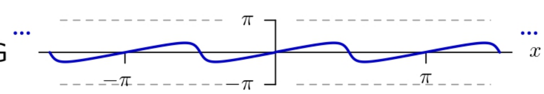
B 

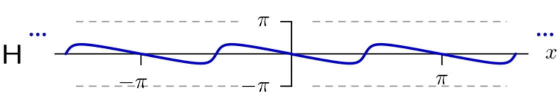
C 

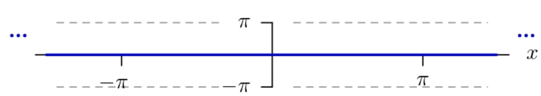
D 

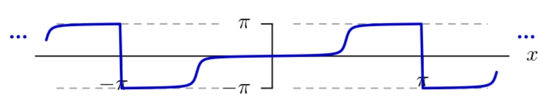
E 

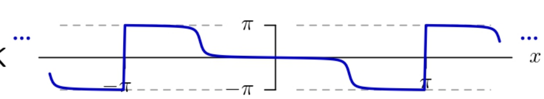
F 

G 

H 

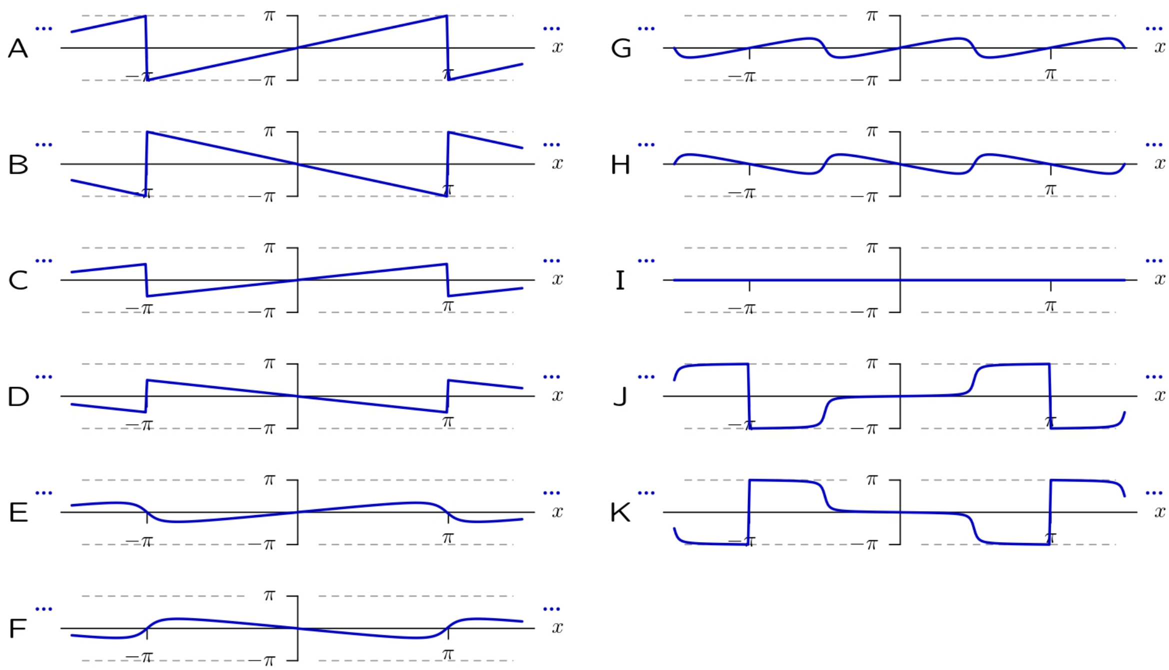
I 

J 

K 

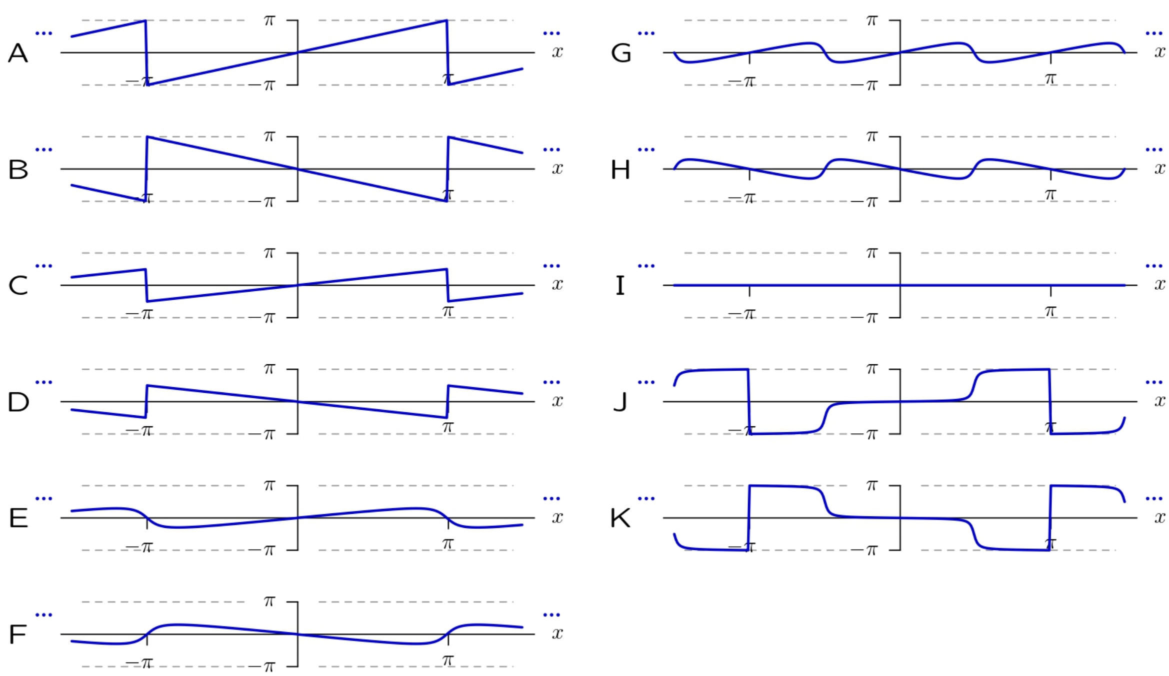
Angular Trends

Which of the following plots shows the angle of $(0.9e^{jx}+0.8e^{-jx})$?



Angular Trends

Which of the following plots shows the angle of $\left(\frac{1}{1+0.8e^{jx}}\right)$?



Discrete-Time Fourier Series: Matrix-Vector Notation

Though we won't use matrix-vector notation in 6.300, some students may appreciate conceptualizing the discrete-time Fourier series analysis and synthesis formulæ as matrix multiplications.

Notation: Let $x_n = x[n]$. So, for example, $x_0 = x[0]$, $x_1 = x[1]$, $x_2 = x[2]$, and so on.

The DTFS analysis formula for a DT signal $x[n]$ which is periodic in $N = 4$ samples

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\underbrace{e^{-j\frac{2\pi}{4}kn}}_{(-j)^{kn}}$$

defines a system of four algebraic equations. Each equation yields a DTFS coefficient.

$$a_0 = \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \qquad a_1 = \frac{1}{4}(x_0 + jx_1 - x_2 - jx_3)$$

$$a_2 = \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \qquad a_3 = \frac{1}{4}(x_0 - jx_1 - x_2 + jx_3)$$

We can write this system of equations in matrix-vector form.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4}\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}_{\text{Fourier analysis matrix}}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Likewise, the DTFS synthesis formula for constructing a periodic DT signal from a linear combination of periodic complex exponential basis functions $e^{jk\Omega_0 n}$

$$x[n] = \sum_{k=0}^{3} a_k \underbrace{e^{j\frac{2\pi}{4}kn}}_{(j)^{kn}}$$

also yields a system of four algebraic equations. Each equation yields a sample of the DT signal $x[n]$.

$$x_0 = a_0 + a_1 + a_2 + a_3 \qquad x_1 = a_0 + ja_1 - a_2 - ja_3$$

$$x_2 = a_0 - a_1 + a_2 - a_3 \qquad x_3 = a_0 - ja_1 - a_2 + ja_3$$

We can write this system of equations in matrix-vector form.

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{a_0\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{DC}} + \underbrace{a_1\begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}}_{\pi/2 \text{ rotations}} + \underbrace{a_2\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}_{\pi \text{ rotations}} + \underbrace{a_3\begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix}}_{3\pi/2 \text{ rotations}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}}_{\text{Fourier synthesis matrix}}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$