# 6.300: Signal Processing

**Tutorial Topic** Sampling
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## From Continuous to Discrete

We refer to the process of discretizing time or space as *sampling*.

$$x[n] = x(n\Delta) \text{ where } n \in \mathbb{Z} \text{ and } \Delta \text{ denotes the sampling period, sampling interval, or time step}$$

We refer to the process of discretizing amplitude as *quantization*. (e.g., rounding)

$$\hat{x}[n] = Q\{x[n]\} \text{ where } Q\{\cdot\} \text{ denotes a quantization operator}$$

$$\text{e.g., } Q_\Delta\{x[n]\} = \Delta \left\lfloor \frac{x[n]}{\Delta} + \frac{1}{2} \right\rfloor \text{ for some constant } \Delta > 0, \text{ where } \lfloor \cdot \rfloor \text{ denotes the floor function}$$

*Digital* signals are discrete in both time and amplitude. Digital systems such as laptops process digital signals. *Discrete-time* signals are discrete in time — but not necessarily in amplitude. In 6.300, we won't study quantization in great depth. We'll focus on *discrete-time signal processing*.

## Sampling and Aliasing

We *sample* a continuous-time signal $x(t)$ every $\Delta$ seconds to obtain a discrete-time signal $x[n]$.

$$x[n] = x(n\Delta) \text{ where } n \in \mathbb{Z} \text{ and } \Delta \text{ denotes the sampling period, sampling interval, or time step}$$

Note that $x[n]$ is a function of the integer $n$, which is enclosed in square brackets. In contrast, $x(t)$ is a function of the real variable $t$, which is enclosed in parentheses. With this notation, $x[n] \neq x(n)$ in general — when you write $x(n)$, you're implicitly saying that $\Delta = 1$.

### Aliasing

Sampling involves throwing away information. If we don't sample frequently enough, the information within our signal will be distorted: Frequencies will "fold in" on each other.

**Nyquist-Shannon sampling theorem:** Let $f_{\max}$ denote the highest frequency in $x(t)$. The minimum sampling rate that prevents aliasing is $2f_{\max}$ — twice the highest frequency in $x(t)$.

## Frequencies

Always keep the dimensions of quantities in mind.

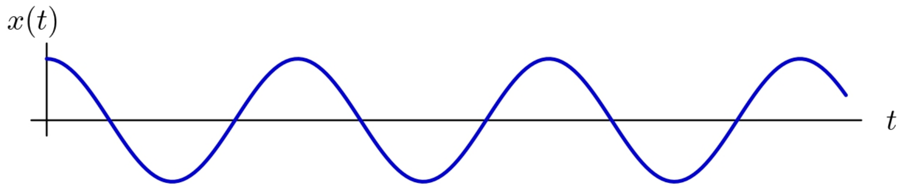
**CT cyclical:** $f = \dfrac{1}{T}$ **CT angular:** $\omega = 2\pi f = \dfrac{2\pi}{T}$ **DT angular:** $\Omega = \dfrac{2\pi f}{f_s} = \dfrac{\omega}{f_s} = \dfrac{2\pi}{N}$

The argument to a trigonometric or exponential function must be expressed in radians.

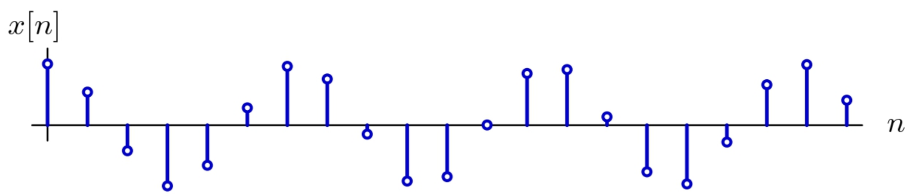
- units$\{2\pi f t\}$ = (radians/cycle) × (cycles/second) × (seconds) = radians
- units$\{\omega t\}$ = (radians/second) × (seconds) = radians
- units$\{\Omega n\}$ = (radians/sample) × (samples) = radians

## Tones and Sinusoids

A "tone" is a pressure that changes sinusoidally with time.




In 6.3000, we will think of this as a "continuous-time" (CT) signal.
In contrast, a "discrete-time" (DT) signal is a sequence of numbers.



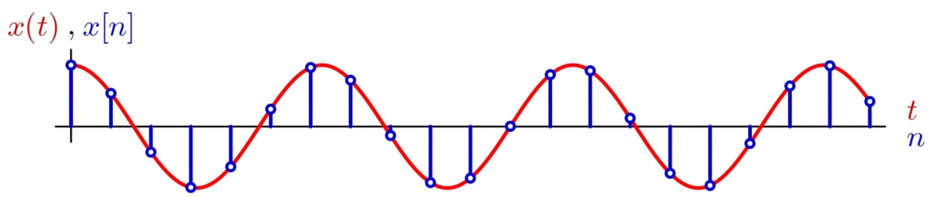

Mathematically:

$$x(t) = A\cos(\omega t)$$

$$x[n] = A\cos(\Omega n)$$

## CT and DT Representations

Assume that $x[n]$ represents "samples" of $x(t)$:




$$x(t) = A\cos(\omega t)$$

$$x[n] = A\cos(\Omega n)$$

- What are the units of $\omega$, $t$, $\Omega$, and $n$?

Let $f$ represent the "frequency" of the tone in cycles/second.

- Determine $\omega$ in terms of $f$.
- Determine $\Omega$ in terms of $\omega$.
- Determine $\Omega$ in terms of $f$.

## Aliases

Compare these two signals.

$$x_1[n] = \cos(3\pi n/4) \qquad\qquad x_2[n] = \cos(5\pi n/4)$$

Which of the following statements are true?

- $x_1[n]$ is periodic in $N = 8$.
- $x_2[n]$ is periodic in $N = 8$.
- $x_1[n] = x_2[n]$ for all $n$.

## Periodicity

Consider the following continuous-time (CT) signal.

$$f(t) = f(t+T) = 6\cos(42\pi t) + 4\cos(18\pi t - \tfrac{1}{2}\pi)$$

What is the fundamental period of this signal? (What is the smallest $T$ such that $f(t) = f(t+T)$ for all $t$?)

## Sampling

Now imagine that we sample $f(t)$ with a sampling rate of $f_s = 60$ samples per second to obtain a discrete-time (DT) signal $f[n]$, which is periodic in $n$ with fundamental period $N$. Determine the discrete-time frequencies (i.e., discrete-time Fourier series coefficients) present in $f[n]$.

## Multiple Periods

The discrete-time signal

$$f[n] = 6\cos(42\pi n/60) + 4\cos(18\pi n/60 - \tfrac{1}{2}\pi)$$

has a fundamental period of $N = 20$ samples. However, this signal is also periodic in $N = 80$. What are the discrete-time Fourier series (DTFS) coefficients if we compute the DTFS of $f[n]$ with $N = 80$?

## Tones in Python

Generate a 1000-hertz tone using a sampling rate of 44100 samples per second. The tone should last 2.5 seconds. Complete the code by determining expressions for `EXPR1` and `EXPR2` below.

```
import math
from lib6300.audio import wav_write
f = [cos(EXPR1 * n) for n in range(0, EXPR2)]
wav_write(f, 44100, 'output.wav')
```

## Optional: Summing Sinusoids

Determine values for $\omega_2$ and $\phi$ such that

$$\sin(\omega_1 t) + \cos(\omega_1 t) = A\cos(\omega_2 t - \phi)$$

for all $t$. You may express your answer in terms of $\omega_1$, $\sin(\cdot)$, $\cos(\cdot)$, $\pi$, $\sqrt{\cdot}$, $j$, and $e$.