# 6.300: Signal Processing

**Tutorial Topic** Sinusoids, Complex Exponentials, and Fourier Series
**Author** Titus K. Roesler (tkr@mit.edu)

**Welcome to tutorial!**
Full attendance is expected. (Don't come if you're sick or there's an emergency, but please let me know.)
We will work at the chalkboards, not on electronic devices. This is not negotiable.
Buddy up! You will work in groups of two. You can switch buddies each tutorial if you want.

## Fourier Series

$f(t) = f(t+T)$ is a $T$-periodic function, and $\omega_0 = 2\pi/T$ denotes the fundamental angular frequency.

### Fourier Series in Trigonometric Form

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t)$$

$$\text{where } c_0 = \frac{1}{T}\int_T f(t)\,dt \text{ and } c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_0 t)\,dt \text{ and } d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_0 t)\,dt$$

$c_0$, the "direct current" (DC) term, represents the average value of $f(t)$ over a single period.

### Fourier Series in Complex Exponential Form

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \text{ where } a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt \qquad \left(\text{e.g., } a_0 = \frac{1}{T}\int_T f(t)dt\right)$$

## Frequency

Time and frequency are inversely proportional.

**cyclical:** $f_0 = \frac{1}{T}$ (cycles per second, or hertz) **angular:** $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$ (radians per second)

## Complex Variables

Think of complex variables geometrically — as points in the complex plane.

$$z = \underbrace{\text{Re}\{z\} + j\,\text{Im}\{z\}}_{\text{rectangular}} = \underbrace{re^{j\phi}}_{\text{polar}} \text{ where } \underbrace{r = \sqrt{\text{Re}\{z\}^2 + \text{Im}\{z\}^2}}_{\text{magnitude of } z} \text{ and } \underbrace{\tan(\phi) = \frac{\text{Im}\{z\}}{\text{Re}\{z\}}}_{\text{angle or phase of } z}$$

### Euler's Formula

Euler's formula relates the rectangular-coordinate and polar-coordinate descriptions of complex variables.

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \cos(\theta) = \text{Re}\{e^{j\theta}\} = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \text{Im}\{e^{j\theta}\} = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

# 6.3000: Signal Processing

## Fourier Series Complex Form

**Synthesis Equation** (making a signal from components):

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

**Analysis Equation** (finding the components)

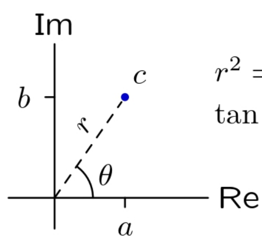
$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

where $\omega_o = \frac{2\pi}{T}$

*February 11, 2025*

## Representations of Complex Numbers

Let $c$ represent a complex number.




$r^2 = a^2 + b^2$
$\tan\theta = \frac{b}{a}$

rectangular form: $c = a + jb$
polar (phasor) form: $r\angle\theta$
Euler form: $r\,e^{j\theta}$

Find

$\angle(jc) - \angle(c)$

which can also be written as

$\arg(jc) - \arg(c)$

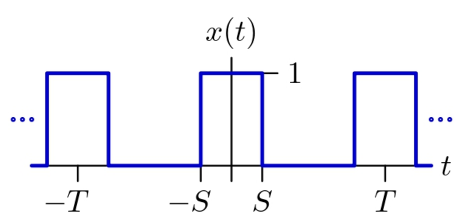
## Complex Numbers

How many of the following are true?

- $\dfrac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
- $(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$
- $|2 + j2 + e^{\frac{j\pi}{4}}| = |2 + j2| + |e^{\frac{j\pi}{4}}|$
- $\mathrm{Im}\left(j^j\right) > \mathrm{Re}\left(j^j\right)$
- $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right) = \tan^{-1} 1$

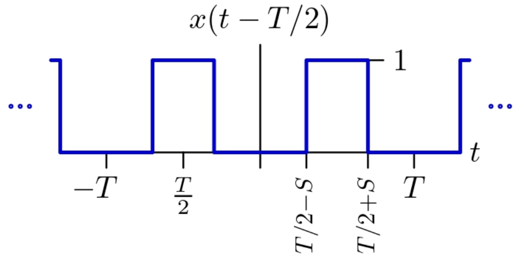
## Pulse Train

Find the Fourier series coefficients $a_k$ for $x(t)$:



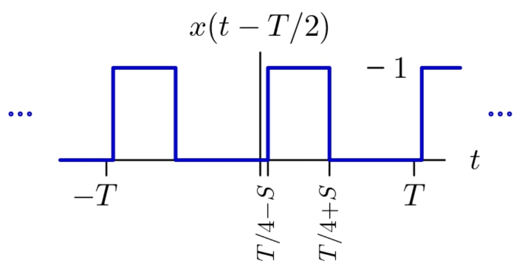
## Pulse Train

What would happen to Fourier series if you delayed $x(t)$ by $T/2$?




## Pulse Train

What would happen if you delayed $x(t)$ by $T/4$?



## Parseval's Theorem

Determine an expression for

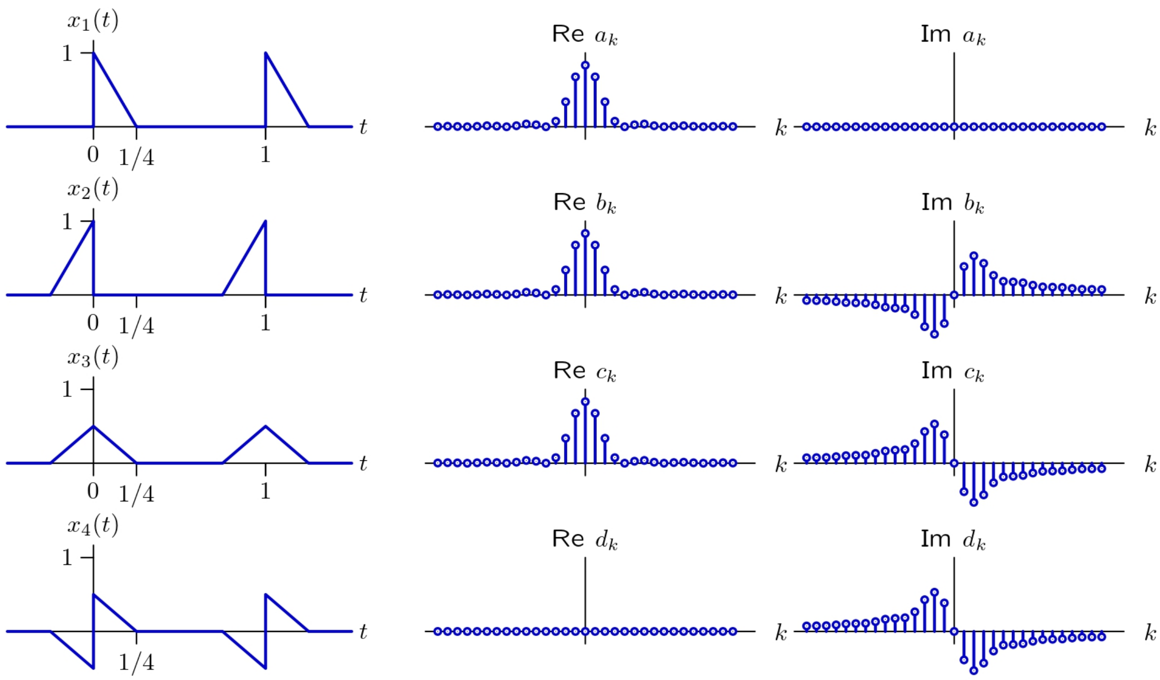
$$\int_T (f(t))^2 dt$$

in terms of the Fourier series coefficients $a_k$ of $f(t)$.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

## Fourier Series Matching

Match the signals (left column) to Fourier series coefficients (right).



## Fourier Series: Measuring Similarity with Sinusoids

This is optional reading — for enrichment. Not required for success in 6.300.

Where did the Fourier analysis and synthesis formulæ come from? Here's a linear-algebra perspective.

Recall: We may represent a $T$-periodic function $f(t)$ as a Fourier series in the trigonometric form

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t) \text{ where } \{c_k\}_{k=0}^{\infty}, \{d_k\}_{k=0}^{\infty} \text{ are Fourier series coefficients}$$

or as a Fourier series in the complex exponential form

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \text{ where } \{a_k\}_{k=-\infty}^{\infty} \text{ are the Fourier series coefficients}$$

where $\omega_0 = 2\pi/T$ denotes the fundamental angular frequency, measured in radians per second.

The Fourier series coefficients — be they $\{c_k\}$ and $\{d_k\}$ or just $\{a_k\}$ — are found by computing inner products (i.e., dot products) between the function $f(t)$ and the sinusoidal basis functions used in the Fourier series expansion. The inner product may be conceptualized as a measure of similarity between a vector $\mathbf{u}$ and a $\mathbf{v}$ — or between a function $f$ and a function $g$.

$$\text{inner product in } L\text{-dimensional vector space: } \langle \mathbf{u} \,|\, \mathbf{v} \rangle = u_1 v_1^* + u_2 v_2^* + \cdots + u_L v_L^* = \sum_{k=1}^{L} u_k v_k^*$$

$$\text{inner product in infinite-dimensional } T\text{-periodic function space: } \langle f \,|\, g \rangle = \int_T f(t) g(t)^* dt$$

Measure the similarity between a $T$-periodic function $f(t)$ and a $(T/k)$-periodic sinusoid using an inner product. This measure of similarity is a function of $k$, the harmonic number.

$$a_k = \underbrace{\langle f(t) \,|\, e^{jk\omega_0 t} \rangle}_{\text{Fourier coefficient}} = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} dt$$

The more similar $f(t)$ and $e^{jk\omega_0 t}$ are, the greater the magnitude of the $k^{\text{th}}$ harmonic coefficient. We reconstruct $f(t)$ as a linear combination of the harmonically-related sinusoidal basis functions.

$$\text{e.g., } \mathbf{u} = u_1 \mathbf{e_1} + u_2 \mathbf{e_2} + \cdots + u_L \mathbf{e_L} \text{ with basis vectors } \{\mathbf{e_i}\}$$

$$\text{e.g., } f(t) = \cdots + a_{-2} e^{-j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \cdots \text{ with basis functions } e^{jk\omega_0 t}$$

A Fourier series is a decomposition of a periodic function into a linear combination of sinusoidal basis functions. $\cos(k\omega_0 t)$ and $\sin(k\omega_0 t)$ are the basis functions used in a trigonometric-form Fourier series, while $e^{jk\omega_0 t}$ are the basis functions used in a complex-form Fourier series.

$$\text{e.g., } \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}}_{\text{signal}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}}_{4\times 4 \text{ Fourier matrix}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}}_{\text{coefficients}} = \underbrace{c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}}_{\text{weighted superposition of Fourier basis vectors}}$$