6.300: Signal Processing

Discrete Cosine Transform

Discrete Cosine Transform

The **discrete cosine transform (DCT)** is defined by analysis and synthesis equations analogous to those of the DFT.

$$X_C[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k}{N}\left(n+\tfrac{1}{2}\right)\right) \qquad \text{(analysis)}$$

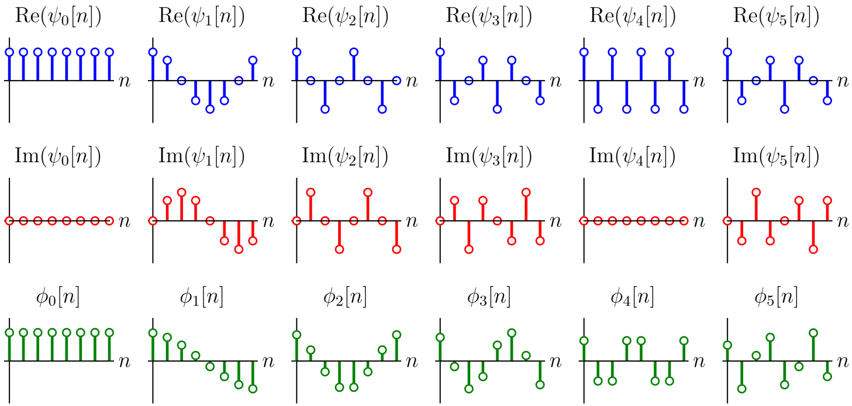
$$x[n] = X_C[0] + 2\sum_{k=1}^{N-1} X_C[k]\cos\left(\frac{\pi k}{N}\left(n+\tfrac{1}{2}\right)\right) \qquad \text{(synthesis)}$$

As described in lecture, the DCT of $x[n]$ is equal to the DFT of $x'[n]$, where $x'[n]$ is derived from $x[n]$ by

- **appending** one period of $x[n]$ in reverse order,
- **stretching** the result in time by a factor of two,
- **doubling** each value, and
- **shifting** the result right by a single sample.

Basis Functions: DFT vs. DCT

The DFT basis functions $\psi_k[n]$ are complex exponentials.
The DCT basis functions $\phi_k[n]$ are real-valued cosines.

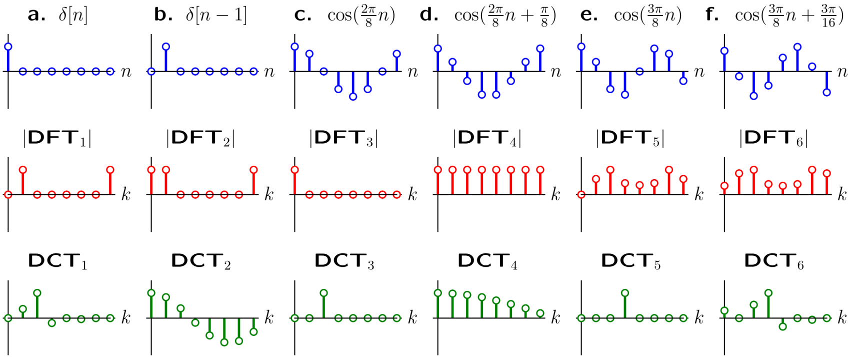


Basis Functions: DFT vs. DCT

For each time-domain signal (**a, b, c, d, e, f**) shown below, determine which of the following plots show

- the 8-point **DFT** coefficient magnitudes and
- the 8-point **DCT** coefficients.

Note that maxima and minima may differ between plots.



Basis Functions: DFT vs. DCT

Much of the utility of Fourier transforms — and, in particular, the DFT — stems from properties of the Fourier basis functions.

DTFS: $e^{jk\frac{2\pi}{N}n}$ **DTFT:** $e^{j\Omega n}$ **DFT:** $e^{jk\frac{2\pi}{N}n}$

To better understand the DCT, we must understand the DCT basis functions.

$$\textbf{DCT: } \cos\left(\frac{\pi k}{N}\left(n+\tfrac{1}{2}\right)\right)$$

DCT Basis Functions: Symmetry

The k^{th} DCT basis function of order N is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \tfrac{1}{2}\right)\right)$$

where $k \in \{0, 1, 2, \ldots, N-1\}$.

How many of the symmetries below are true?

- $\phi_k[n + 2N] = \phi_k[n]$
- $\phi_k[n + N] = (-1)^k \phi_k[n]$
- $\phi_k[n - N] = (-1)^k \phi_k[n]$
- $\phi_k[(N-1) - n] = (-1)^k \phi_k[n]$

DCT Basis Functions: Area

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] = N\delta[k] = \begin{cases} N & k = 0 \\ 0 & k \neq 0. \end{cases}$$

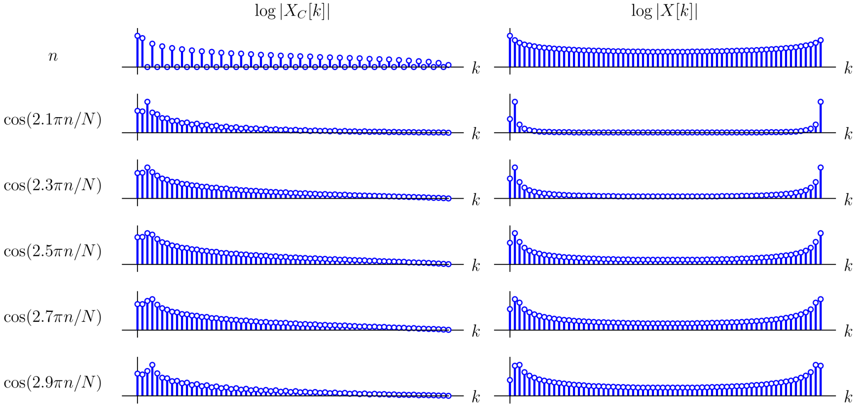
DCT Basis Functions: Orthogonality

The DCT basis functions are orthogonal. Show that

$$\sum_{n=0}^{N-1} \phi_k[n]\phi_\ell[n] = \begin{cases} N & k = \ell = 0 \\ \frac{1}{2}N & k = \ell \neq 0 \\ 0 & k \neq \ell. \end{cases}$$

DFT vs. DCT: Compaction

If a signal has predominately low-frequency content, then the higher-order coefficients of the DCT tend to decrease faster than the corresponding coefficients of the DFT. Similar compaction results for sinusoidal signals.



DFT vs. DCT: Eigenfunctions

The Fourier basis functions $e^{j\Omega n}$ are **eigenfunctions** of linear, time-invariant systems. The **eigenvalues** are $H(\Omega)$.

$$e^{j\Omega n} \rightarrow \boxed{\text{LTI}} \rightarrow H(\Omega) e^{j\Omega n}$$

The DCT basis functions are not eigenfunctions of linear, time-invariant systems, however. So, the DCT cannot be used for filtering.