6.300: Signal Processing

Two-Dimensional Fourier Transforms

2D DFT: Analysis and Synthesis Equations

$F[k_c, k_c] = \frac{1}{RC} \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} f[r, c] e^{-j(k_r \frac{2\pi}{R} r + k_c \frac{2\pi}{C} c)}$

$f[r, c] = \sum_{k_r=0}^{R-1} \sum_{k_c=0}^{C-1} F[k_r, k_c] e^{j(k_r \frac{2\pi}{R} r + k_c \frac{2\pi}{C} c)}$

Separability: $f[r, c] = f_{\mathcal{R}}[r] f_{\mathcal{C}}[c] \iff F_{\mathcal{R}}[k_r] F_{\mathcal{C}}[k_c]$

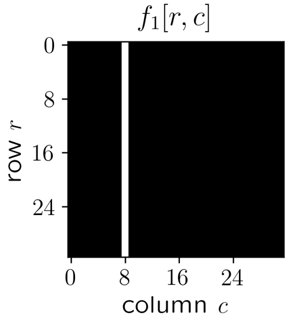
The 2D DFT of a vertical bar is a horizontal bar — and vice versa. The 2D DFT of an impulse train is an impulse train with inversely-proportional spacing.

Agenda for Recitation

- Two-dimensional discrete Fourier transform (2D DFT)

What questions do you have from lecture?

2D Discrete Fourier Transform



Consider the 2D signal $f_1[r, c]$ shown above. Assume that black corresponds to a value of zero and white corresponds to a value of one. Determine an expression for $F_1[k_r, k_c]$, the 2D DFT of $f_1[r, c]$.

2D Discrete Fourier Transform

Determine an expression for $f_1[r,c]$ first.

$$f_1[r,c] = \delta[c-8]$$

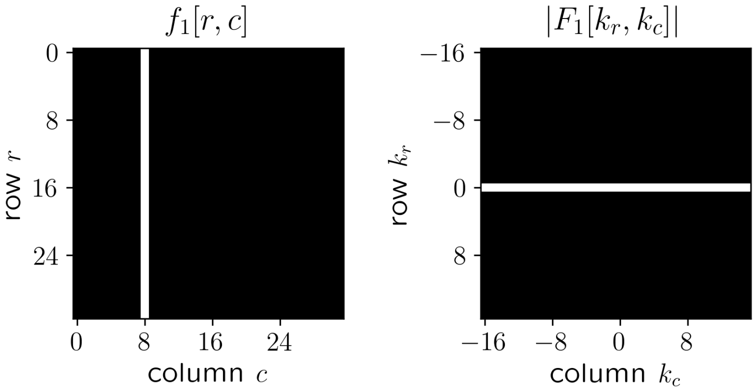
Express $f_1[r,c]$ as the product of $f_{\mathcal{R}}[r]$ and $f_{\mathcal{C}}[c]$.

$$f_{\mathcal{R}}[r] = 1 \iff F_{\mathcal{R}}[k_r] = \delta[k_r]$$
$$f_{\mathcal{C}}[c] = \delta[c-8] \iff F_{\mathcal{C}}[k_c] = \frac{1}{32} e^{-jk_c \frac{2\pi}{32} 8}$$

The transform $F_1[k_r,k_c]$ is the product of $F_{\mathcal{R}}[k_r]$ and $F_{\mathcal{C}}[k_c]$.

$$F_1[k_r,k_c] = \frac{1}{32}\delta[k_r] e^{-jk_c \frac{2\pi}{32} 8} = \begin{cases} \frac{1}{32} e^{-jk_c \frac{2\pi}{32} 8} & k_r = 0 \\ 0 & k_r \neq 0 \end{cases}$$

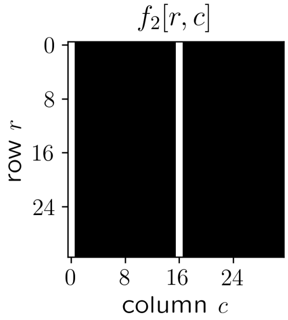
2D Discrete Fourier Transform



The transform of a vertical bar is a horizontal bar.

- Could you change $f_1[r,c]$ so that $F_1[k_r,k_c]=\frac{1}{32}\delta[k_r]$?
- Could you change $f_1[r,c]$ so that $|F_1[k_r,k_c]|=\frac{1}{32}\delta[k_r-8]$?

2D Discrete Fourier Transform



Consider the 2D signal $f_2[r,c]$ shown above. Assume that black corresponds to a value of zero and white corresponds to a value of one. Determine an expression for $F_2[k_r,k_c]$, the 2D DFT of $f_2[r,c]$.

2D Discrete Fourier Transform

Determine an expression for $f_2[r,c]$ first.

$$f_2[r,c] = \delta[c] + \delta[c-16]$$

Express $f_2[r,c]$ as the product of $f_{\mathcal{R}}[r]$ and $f_{\mathcal{C}}[c]$.

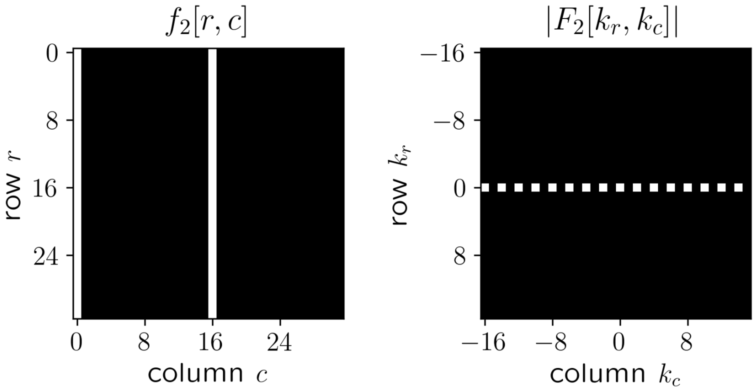
$$f_{\mathcal{R}}[r] = 1 \iff F_{\mathcal{R}}[k_r] = \delta[k_r]$$

$$f_{\mathcal{C}}[c] = \delta[c] + \delta[c-16] \iff F_{\mathcal{C}}[k_c] = \frac{1}{32} + \frac{1}{32} e^{-jk_c \frac{2\pi}{32} 16} = \frac{1+(-1)^{k_c}}{32}$$

The transform $F_2[k_r,k_c]$ is the product of $F_{\mathcal{R}}[k_r]$ and $F_{\mathcal{C}}[k_c]$.

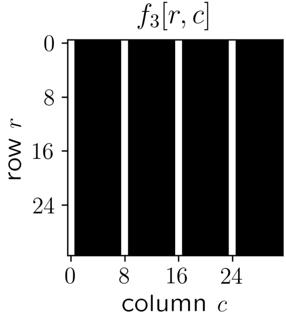
$$F_2[k_r,k_c] = \delta[k_r]\frac{1+(-1)^{k_c}}{32} = \begin{cases} \frac{1}{16} & k_r = 0 \text{ **and** } k_c \text{ even} \\ 0 & k_r \neq 0 \text{ **or** } k_c \text{ odd} \end{cases}$$

2D Discrete Fourier Transform



The transform of an impulse train is an impulse train with inversely-proportional spacing.

2D Discrete Fourier Transform



Consider the 2D signal $f_3[r, c]$ shown above. Assume that black corresponds to a value of zero and white corresponds to a value of one. Determine an expression for $F_3[k_r, k_c]$, the 2D DFT of $f_3[r, c]$.

2D Discrete Fourier Transform

Determine an expression for $f_3[r,c]$ first.

$$f_3[r,c] = \delta[c] + \delta[c-8] + \delta[c-16] + \delta[c-24]$$

Express $f_3[r,c]$ as the product of $f_{\mathcal{R}}[r]$ and $f_{\mathcal{C}}[c]$.

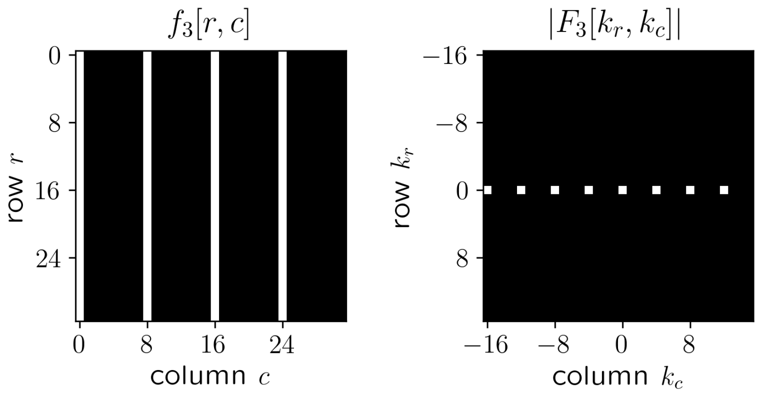
$$f_{\mathcal{R}}[r] = 1 \iff F_{\mathcal{R}}[k_r] = \delta[k_r]$$

$$f_{\mathcal{C}}[c] = \sum_{m=0}^{3} \delta[c-8m] \iff F_{\mathcal{C}}[k_c] = \frac{1 + (-j)^{k_c} + (-1)^{k_c} + j^{k_c}}{32}$$

The transform $F_3[k_r,k_c]$ is the product of $F_{\mathcal{R}}[k_r]$ and $F_{\mathcal{C}}[k_c]$.

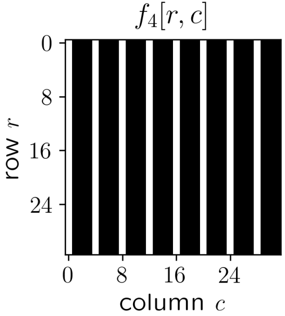
$$\begin{aligned} F_3[k_r,k_c] &= \delta[k_r]\frac{1 + (-j)^{k_c} + (-1)^{k_c} + j^{k_c}}{32} \\ &= \begin{cases} \frac{1}{8} & k_r = 0 \text{ **and** } k_c \bmod 4 = 0 \\ 0 & k_r \neq 0 \text{ **or** } k_c \bmod 4 \neq 0 \end{cases} \end{aligned}$$

2D Discrete Fourier Transform



The transform of an impulse train is an impulse train with inversely-proportional spacing.

2D Discrete Fourier Transform



Consider the 2D signal $f_4[r, c]$ shown above. Assume that black corresponds to a value of zero and white corresponds to a value of one. Determine an expression for $F_4[k_r, k_c]$, the 2D DFT of $f_4[r, c]$.

2D Discrete Fourier Transform

Determine an expression for $f_4[r,c]$ first.

$$f_4[r,c] = \sum_{m=0}^{7} \delta[c-4m]$$

Express $f_4[r,c]$ as the product of $f_{\mathcal{R}}[r]$ and $f_{\mathcal{C}}[c]$.

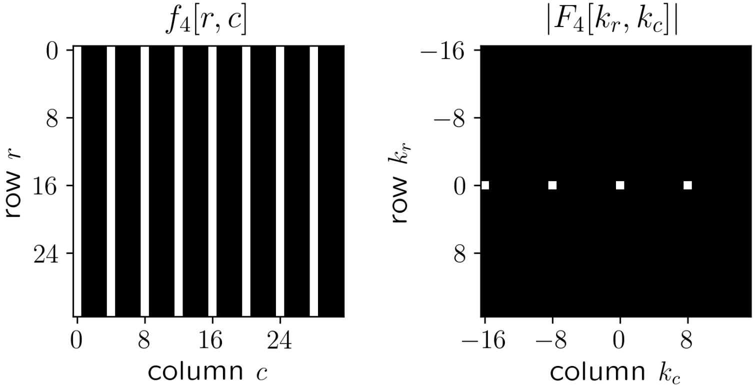
$$f_{\mathcal{R}}[r] = 1 \iff F_{\mathcal{R}}[k_r] = \delta[k_r]$$

$$f_{\mathcal{C}}[c] = \sum_{m=0}^{7} \delta[c-4m] \iff F_{\mathcal{C}}[k_c] = \begin{cases} \frac{1}{4} & k_c \bmod 8 = 0 \\ 0 & k_c \bmod 8 \neq 0 \end{cases}$$

The transform $F_4[k_r,k_c]$ is the product of $F_{\mathcal{R}}[k_r]$ and $F_{\mathcal{C}}[k_c]$.

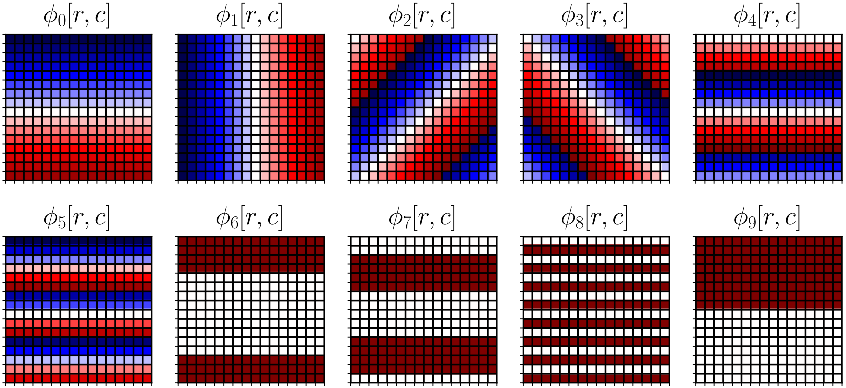
$$F_4[k_r,k_c] = \begin{cases} \frac{1}{4} & k_r = 0 \textbf{ and } k_c \bmod 8 = 0 \\ 0 & k_r = 0 \textbf{ or } k_c \bmod 8 \neq 0 \end{cases}$$

2D Discrete Fourier Transform



The transform of an impulse train is an impulse train with inversely-proportional spacing.

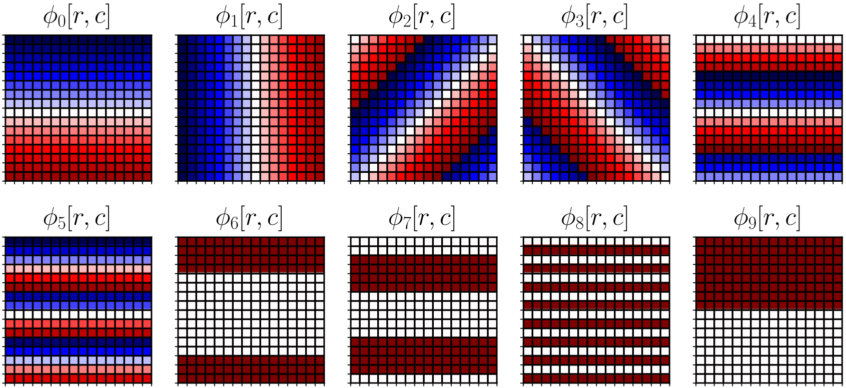
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_0.

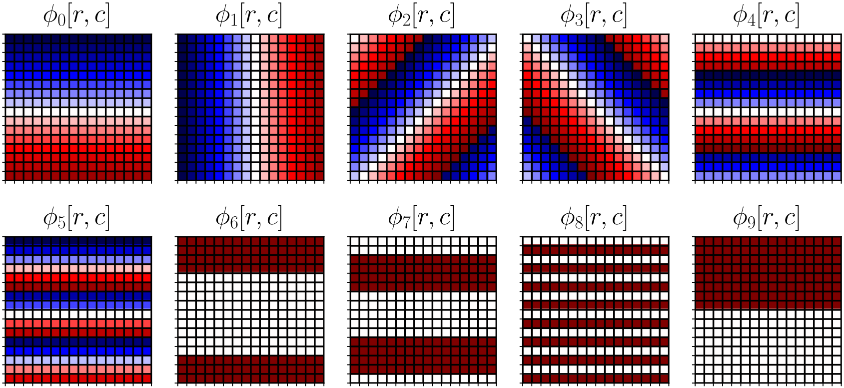
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_0. $e^{j\frac{2\pi}{R}r}$

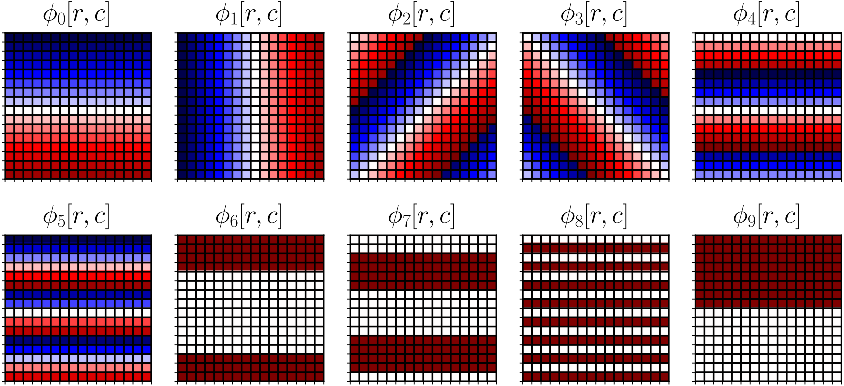
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_1.

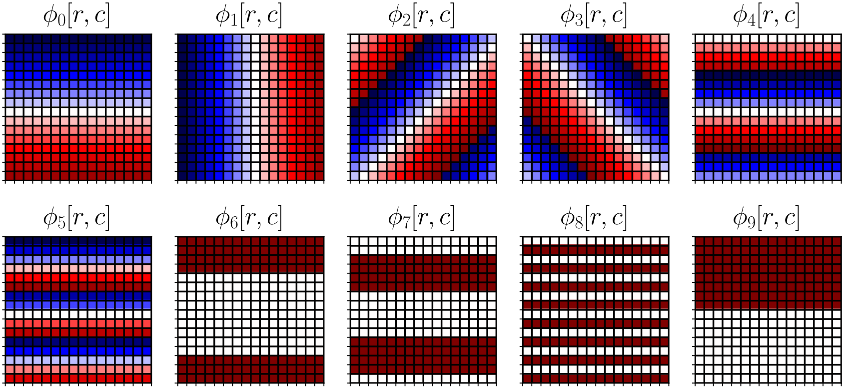
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_1. $e^{j\frac{2\pi}{C}c}$

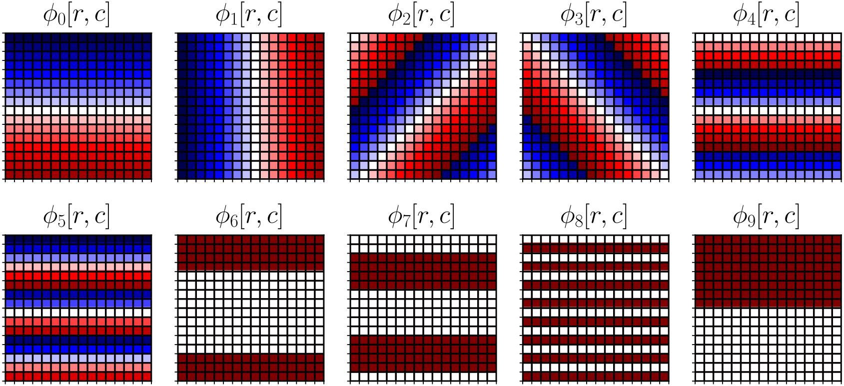
Plane Waves



The origin $(0, 0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_2.

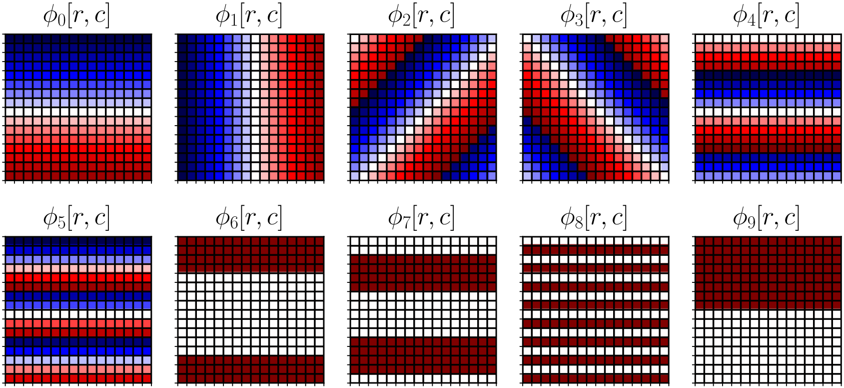
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel.
Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_2. $e^{j(\frac{2\pi}{R}r+\frac{2\pi}{C}c)}$

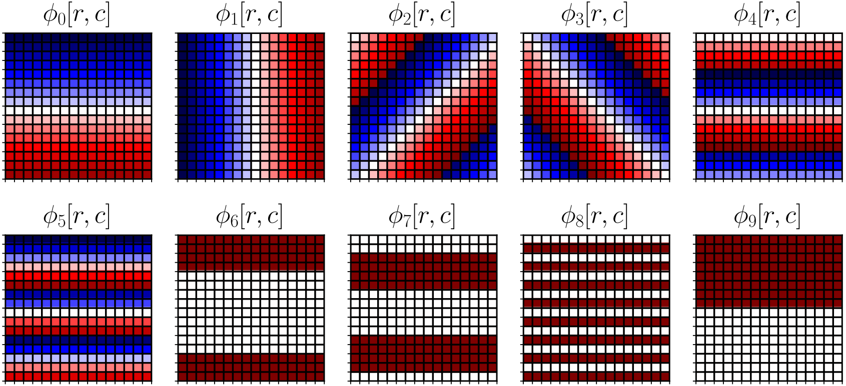
Plane Waves



The origin $(0, 0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_3.

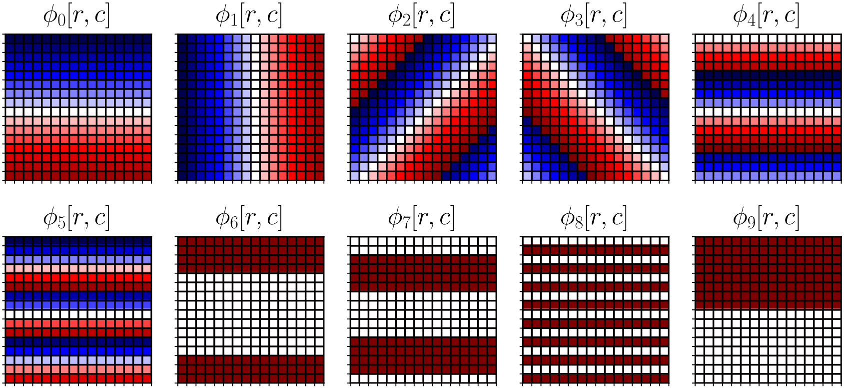
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_3. $e^{j(\frac{2\pi}{R}r - \frac{2\pi}{C}c)}$

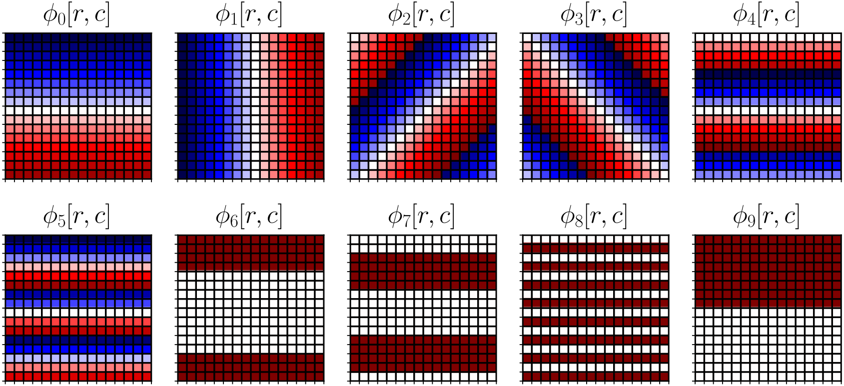
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_4.

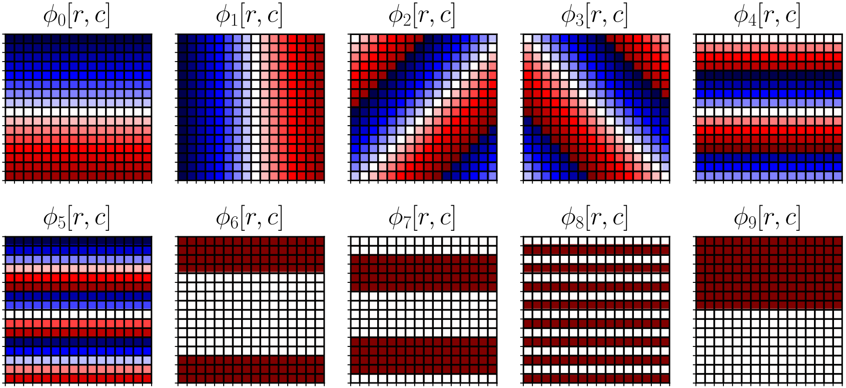
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_4. $e^{j\frac{4\pi}{R}r}$

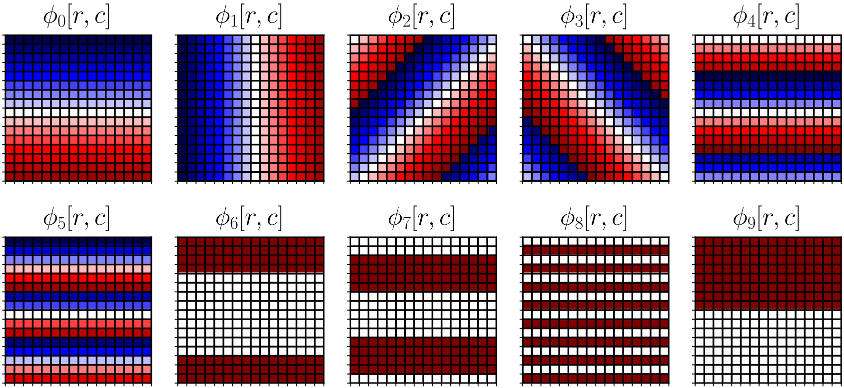
Plane Waves



The origin $(0, 0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_5.

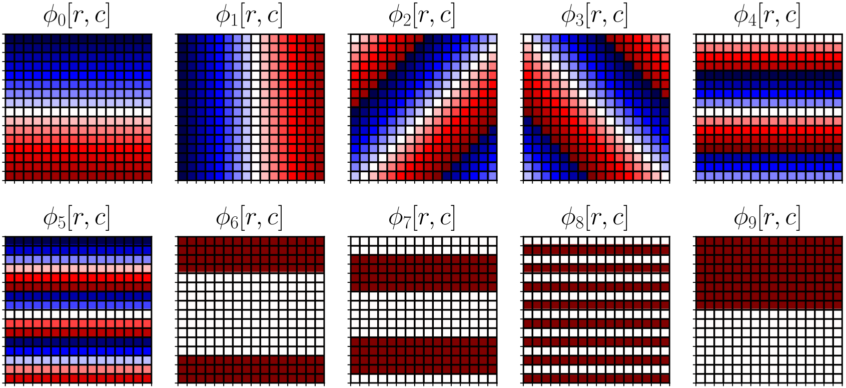
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_5. $e^{j\frac{6\pi}{R}r}$

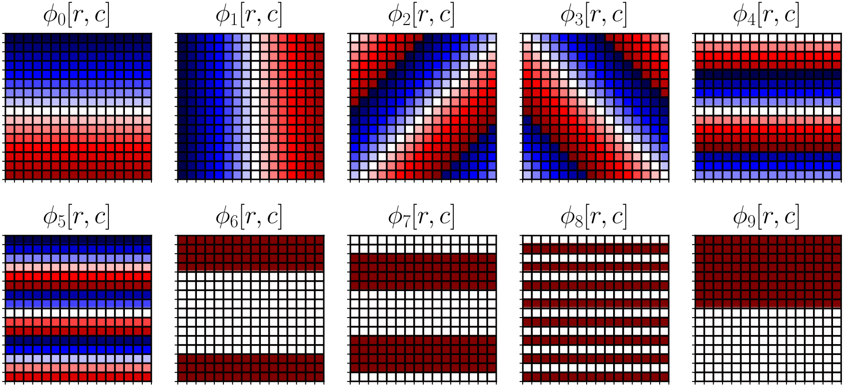
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_6.

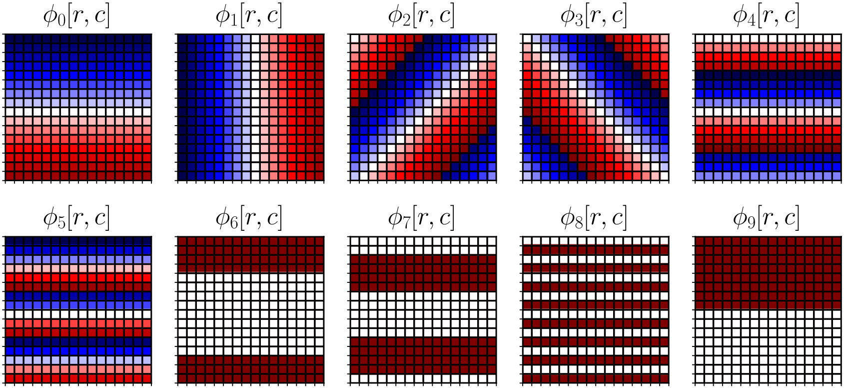
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_6. $\cos\left(\frac{2\pi}{R} r\right)$

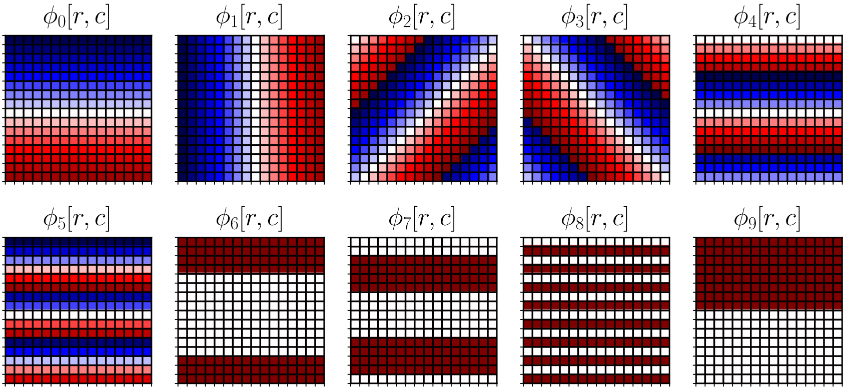
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_7.

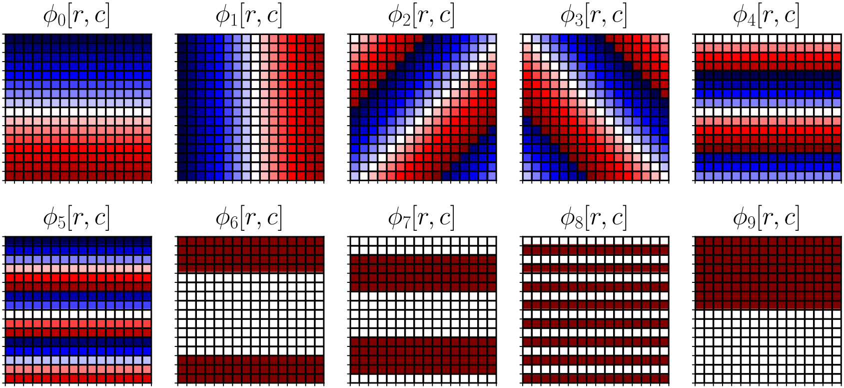
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_7. $\cos\left(\frac{4\pi}{R} r\right)$

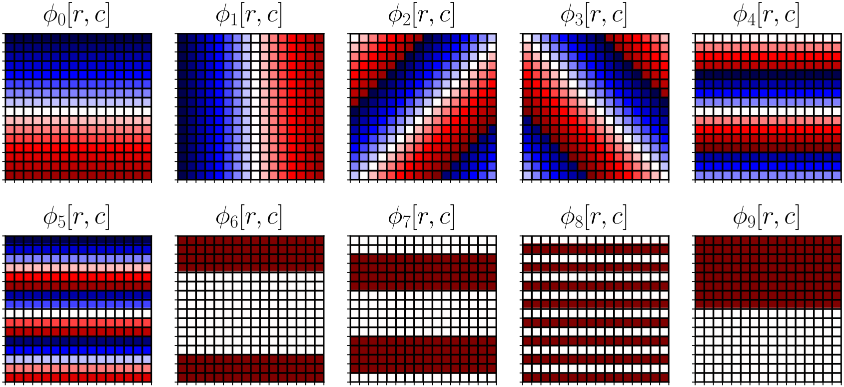
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_8.

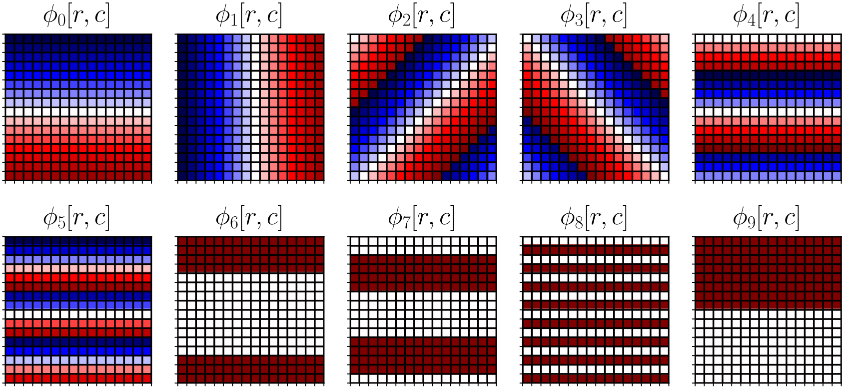
Plane Waves



The origin $(0, 0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_8. $\cos(\pi r) = (-1)^r$

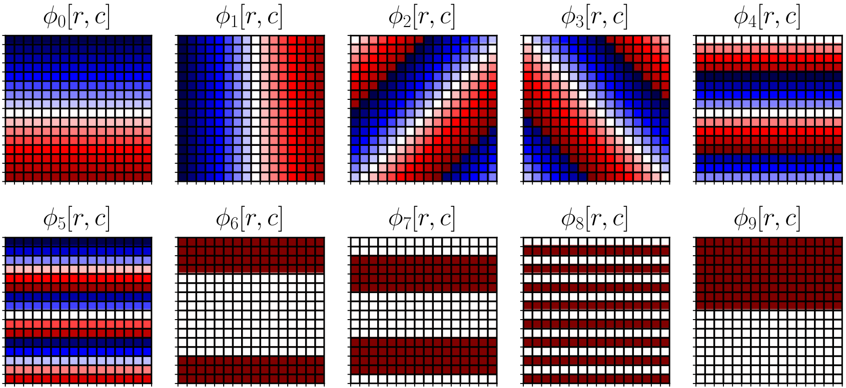
Plane Waves



The origin $(0,0)$ lies at the center of each $R \times C$ panel. Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_9.

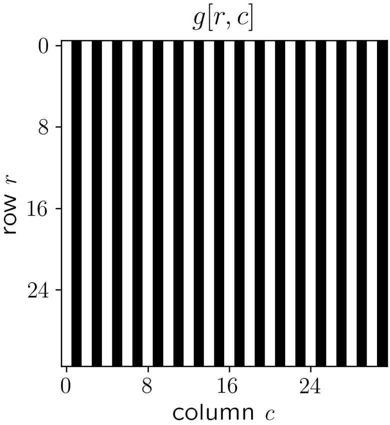
Plane Waves



The origin $(0, 0)$ lies at the center of each $R \times C$ panel.
Blue denotes $-\pi$, white denotes zero, and red denotes π.

Determine a waveform with angle ϕ_9. $\sin\left(\frac{2\pi}{R} r\right)$

Question of the Day



Determine the 2D DFT of $g[r,c]$.