

6.300: Signal Processing

Short-Time Fourier Transform

Windows: Consider the main lobe and sidelobes.

Time Resolution vs. Frequency Resolution

- time resolution: short $w[n] \iff$ wide $W(\Omega)$
- frequency resolution: long $w[n] \iff$ narrow $W(\Omega)$

Analysis:
$$X[k, m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n + ms] w[n] e^{-jk \frac{2\pi}{N} n}$$

Spectrogram: $|X[k, m]|^2$ (magnitude² of STFT)

Agenda for Recitation

- Windows: Rectangular, triangular, and Hann
- Short-time Fourier transforms and spectrograms
- Examples of spectrograms (as time allows)

What questions do you have from lecture?

Windows

Multiplying $x[n]$ by the window function $w[n]$ corresponds to convolving the DTFT of $x[n]$ with the DTFT of $w[n]$.

Windowing

$$x_w[n] = x[n]w[n] \iff X_w(\Omega) = \frac{1}{2\pi}(X * W)(\Omega)$$

- time resolution: short $w[n]$ $\iff$ wide $W(\Omega)$
- frequency resolution: long $w[n]$ $\iff$ narrow $W(\Omega)$

There is a **trade-off** between time and frequency resolution. This is Heisenberg's uncertainty principle!

Windows

There are many window functions. SciPy has 25 of them!

scipy.signal.windows: Bartlett, Bartlett-Hann, Blackman, Blackman-Harris, Bohman, boxcar, cosine, discrete prolate spheroidal sequences, Dolph-Chebyshev, exponential, flat-top, Gaussian, generalized cosine, generalized Gaussian, generalized Hamming, Hamming, Hann, Kaiser, Kaiser-Bessel, Lanczos, Nutall, Parzen, Taylor, triangular, Tukey

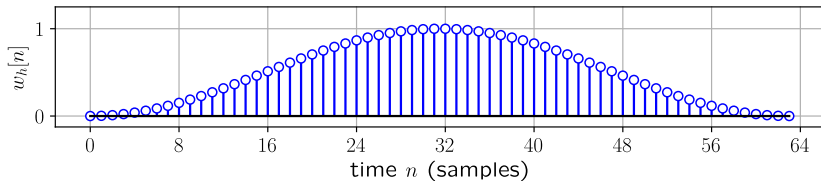
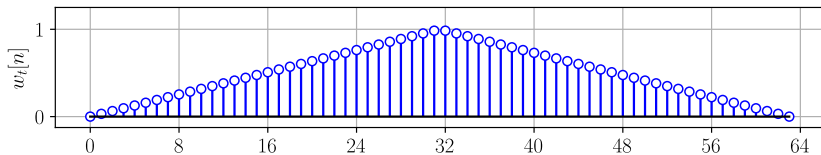
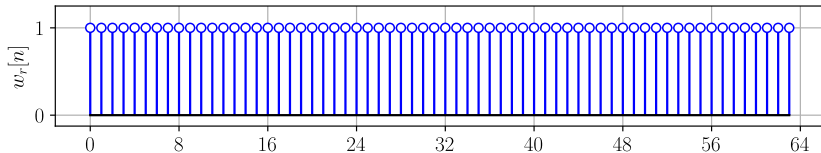
Let's examine the DTFT of a few windows.

- **rectangular:** narrow mainlobe, high sidelobes
- **triangular:** wider mainlobe, shorter sidelobes
- **Hann:** even wider mainlobe, even shorter sidelobes

Good frequency resolution means “narrow mainlobe.”
Problem: Sidelobes may obscure low-power signals.

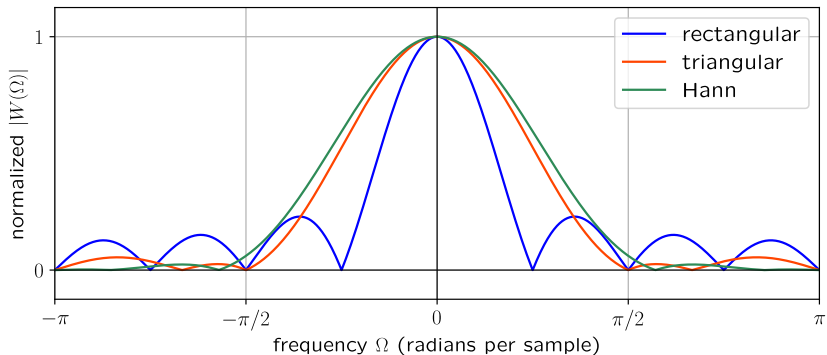
Windows

$N = 64$ samples



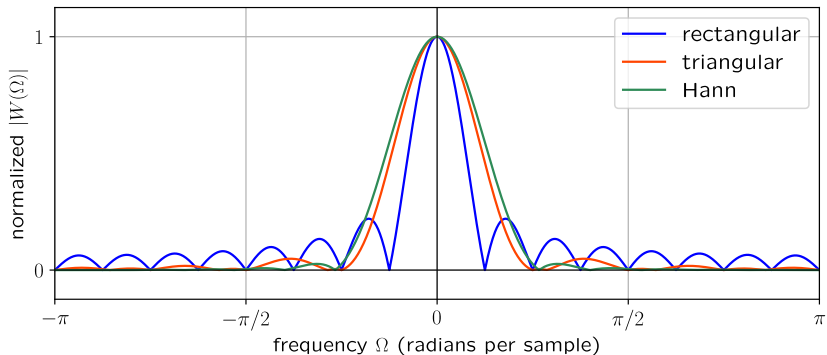
Windows

window comparison ($N = 8$ samples)



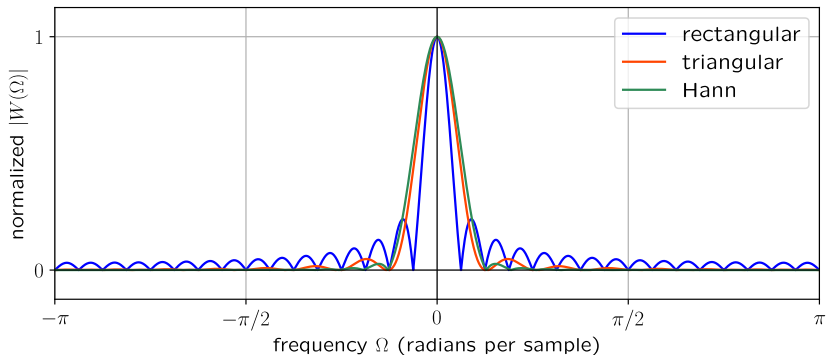
Windows

window comparison ($N = 16$ samples)



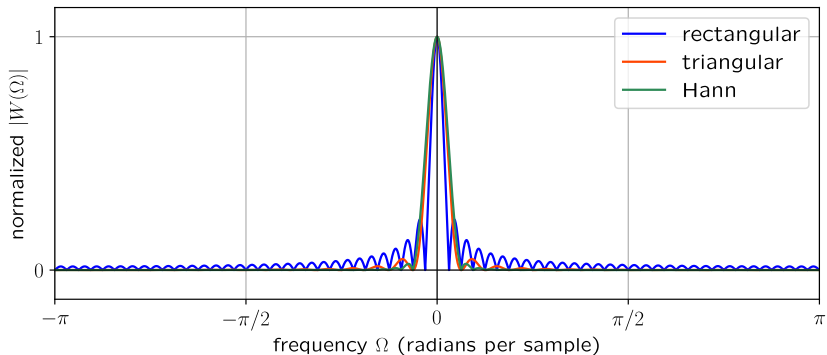
Windows

window comparison ($N = 32$ samples)



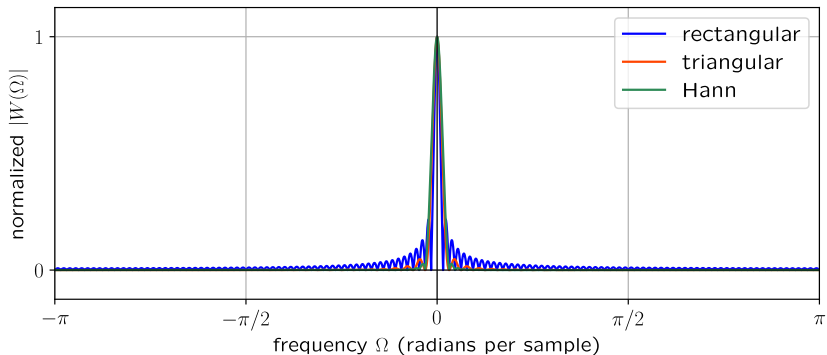
Windows

window comparison ($N = 64$ samples)



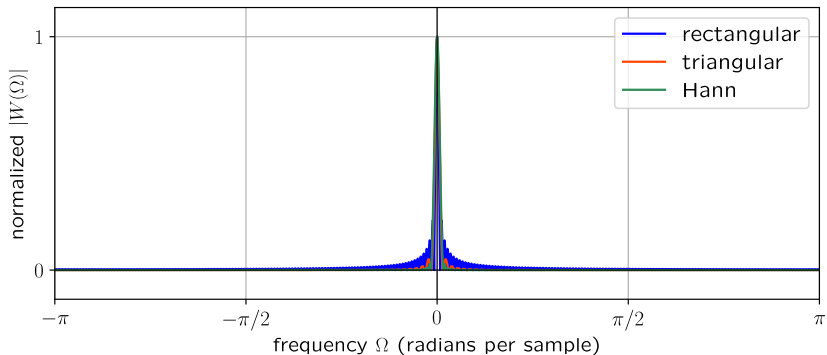
Windows

window comparison ($N = 128$ samples)



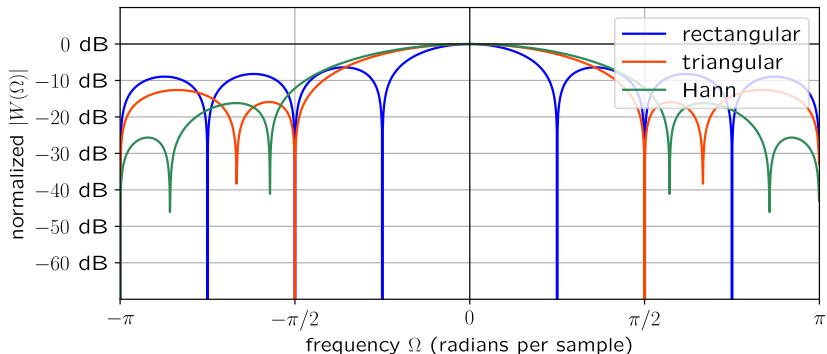
Windows

window comparison ($N = 256$ samples)



Windows

window comparison ($N = 8$ samples)

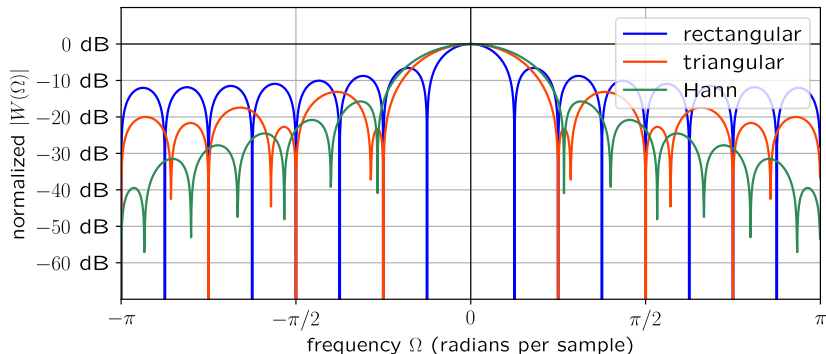


Decibels: $x_{\text{dB}} = 10 \log_{10}(x)$

e.g., $-10 \text{ dB} \iff x = 10^{-1}$ and $-20 \text{ dB} \iff x = 10^{-2}$

Windows

window comparison ($N = 16$ samples)

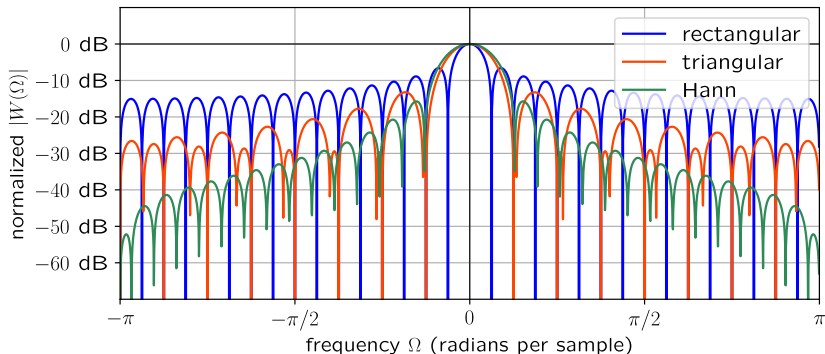


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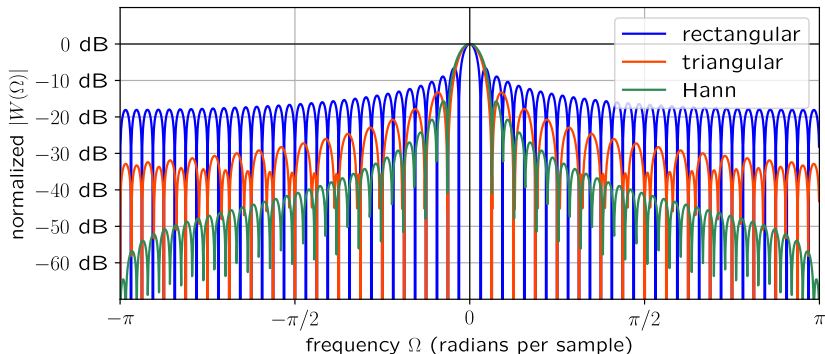


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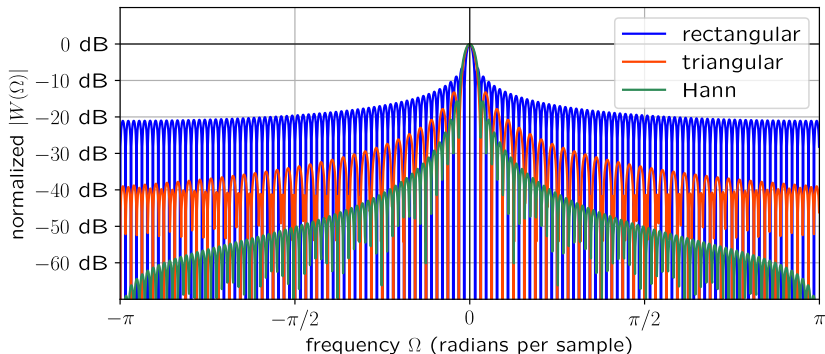


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window comparison ($N = 128$ samples)

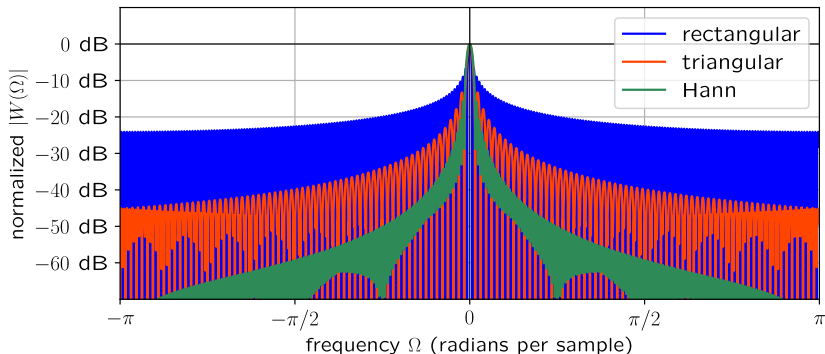


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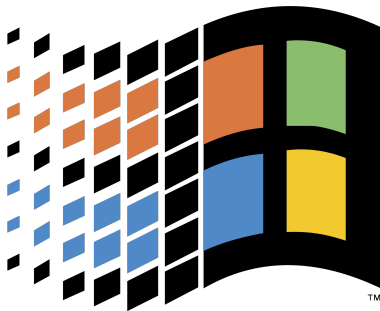
window comparison ($N = 256$ samples)



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Microsoft Windows



MICROSOFT[®]
WINDOWS[™]

Warning!

Your computer may crash if you use Microsoft Windows.

Short-Time Fourier Transform

STFT: Analyze a signal “block by block.” An STFT is a sequence of windowed Fourier transforms.

$$X[k, m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n + ms] w[n] e^{-jk \frac{2\pi}{N} n}$$

Spectrogram: $|X[k, m]|^2$

Parameters:

- type of window: $w[n]$
- DFT size: N
- step size: s

Short-Time Fourier Transform

Parameters:

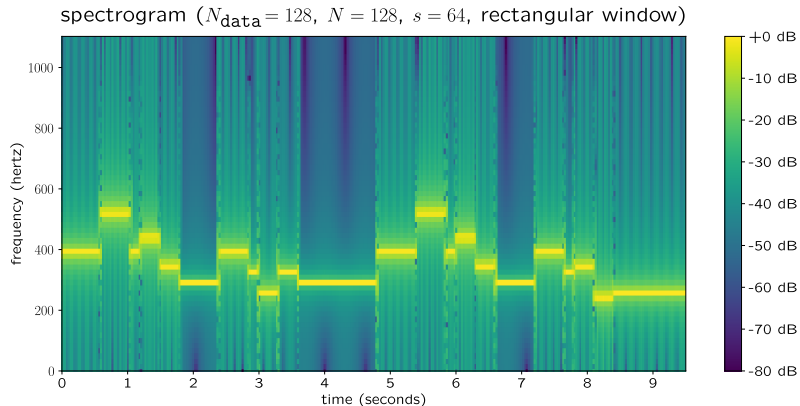
- type of window: $w[n]$
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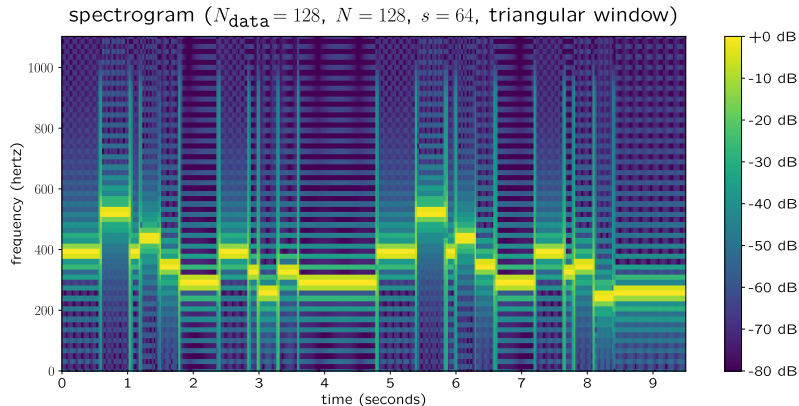
STFT: Windows



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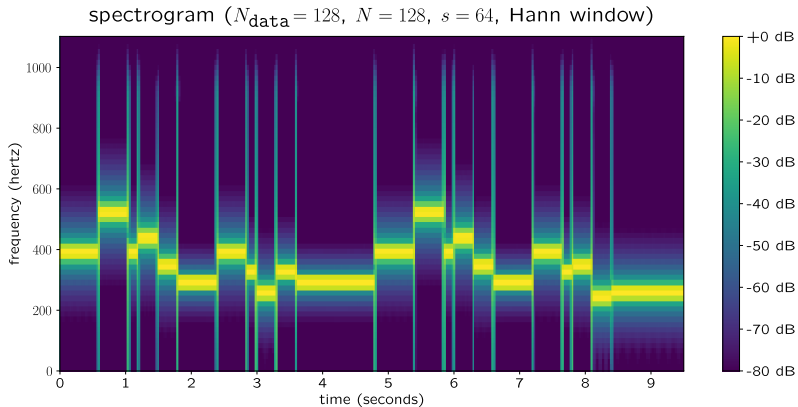
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Short-Time Fourier Transform

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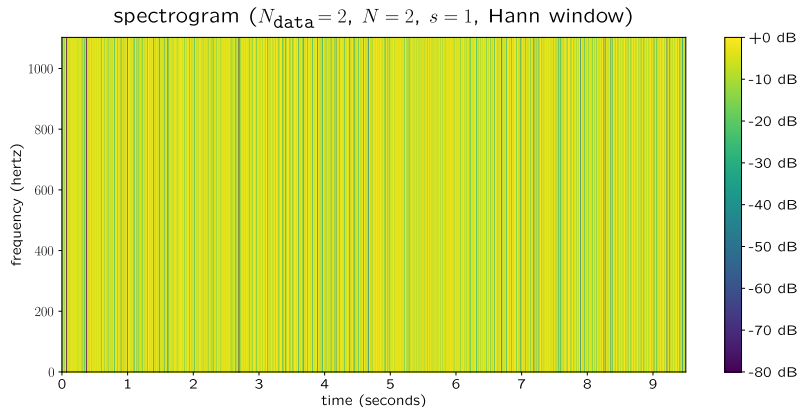
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Short-Time Fourier Transform

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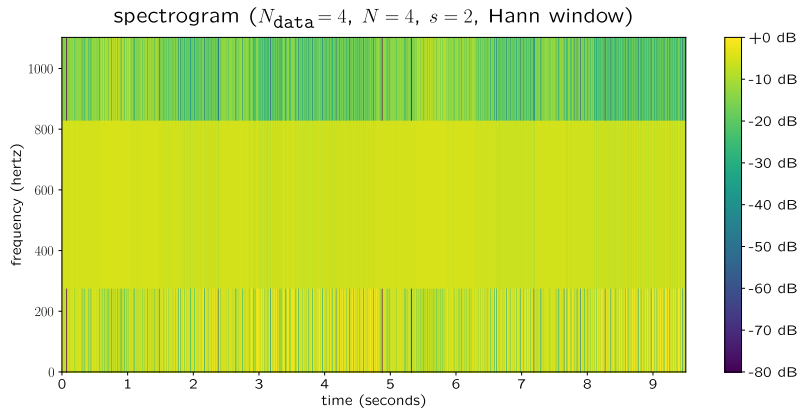
STFT: DFT Size



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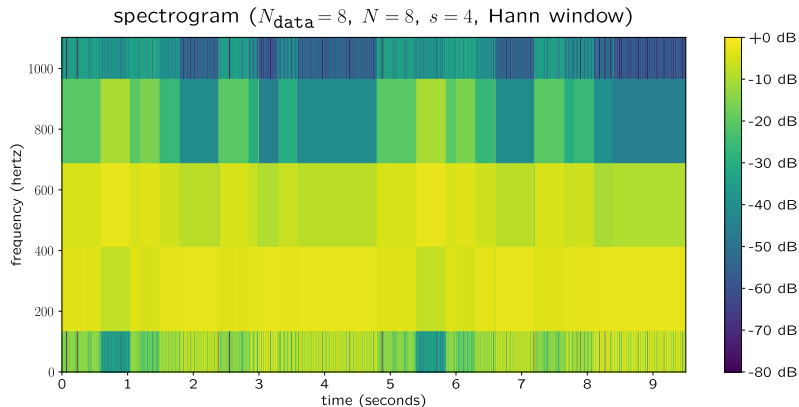
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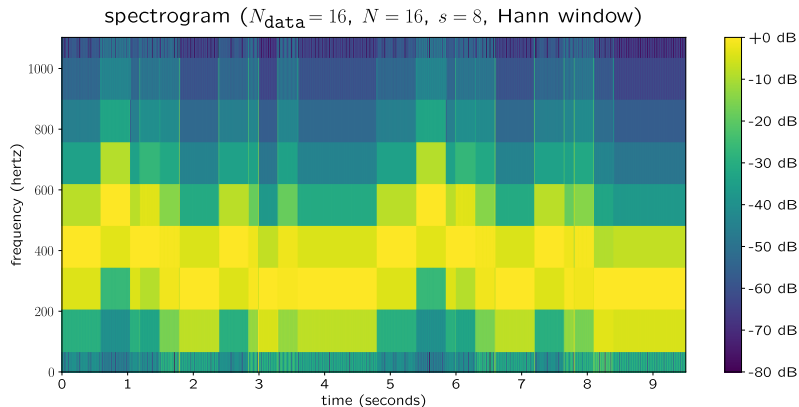
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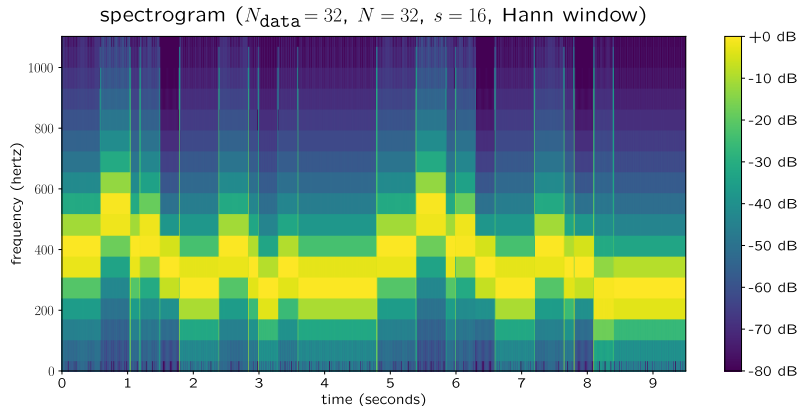
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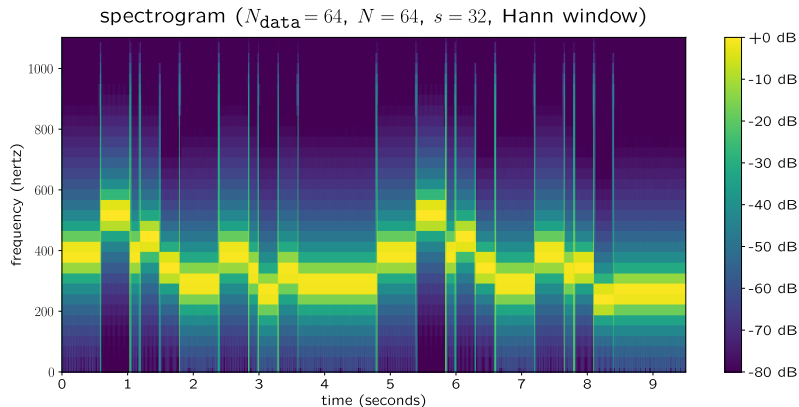
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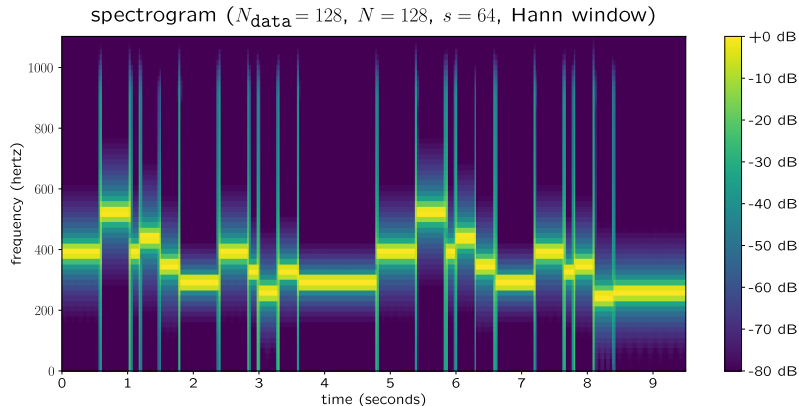
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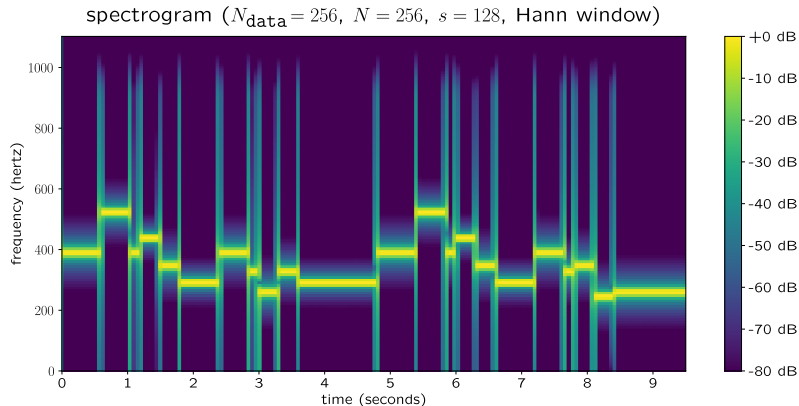
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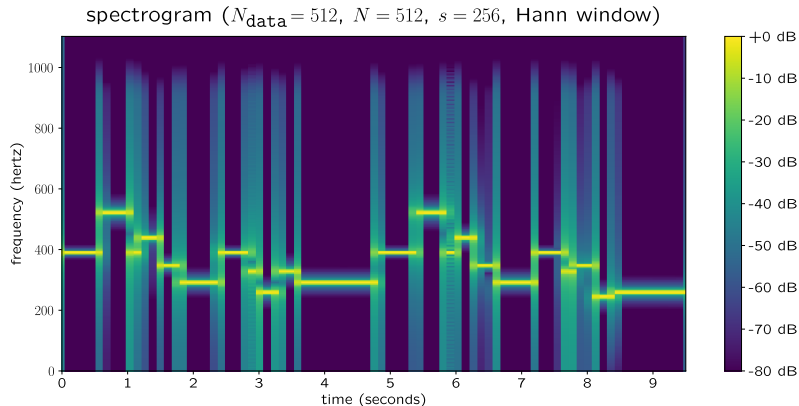
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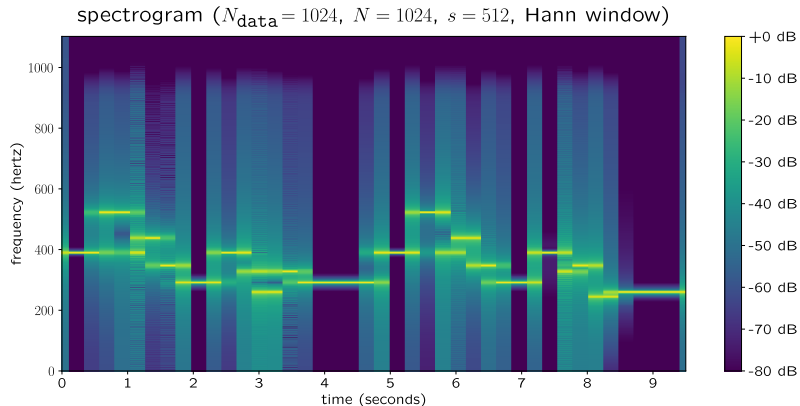
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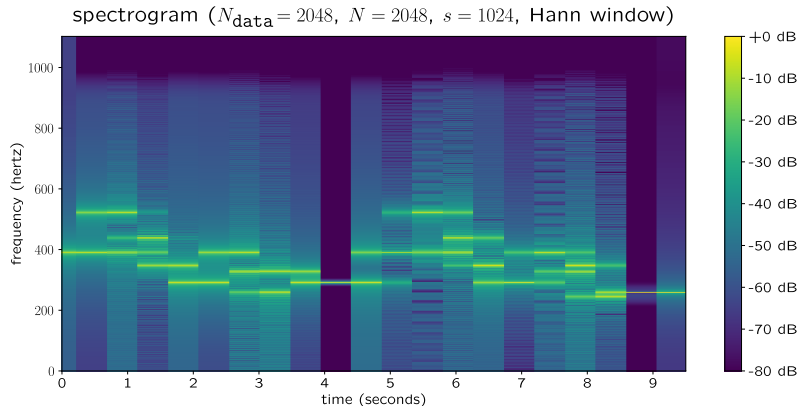
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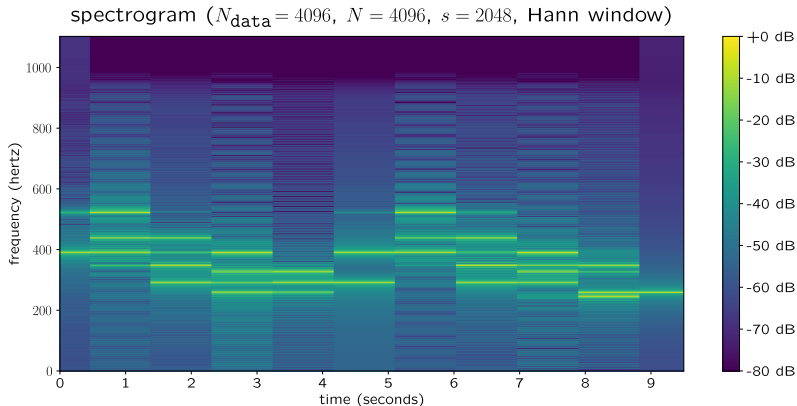
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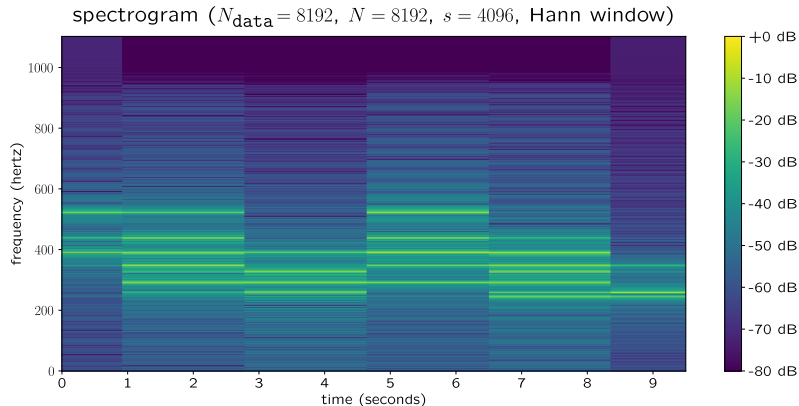
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STFT: Zero-Padding

Suppose that ...

- N_{data} is the length of the data.
- N is the DFT size.

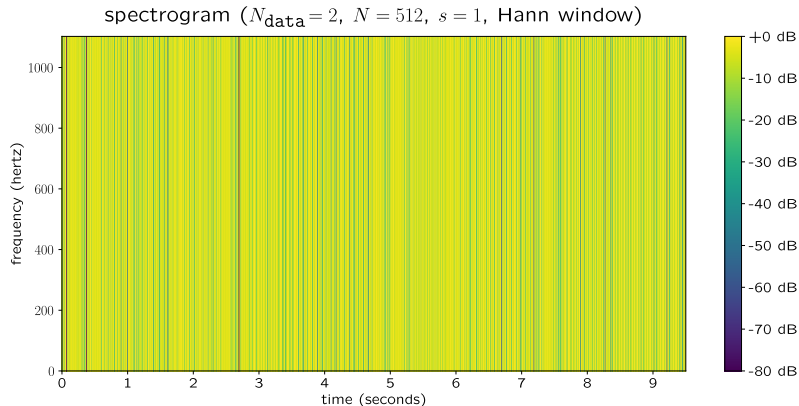
When $N > N_{\text{data}}$, we assume that we are **zero-padding**.

As N increases, we sample the underlying DTFT at more points — but this does not alter $W(\Omega)$, the DTFT of the window function. The width of $W(\Omega)$'s mainlobe ($\sim 1/N_{\text{data}}$) determines frequency resolution.

The more data we have, the larger N_{data} can be. As N_{data} increases, $W(\Omega)$ gets narrower — and the frequency resolution improves.

You cannot improve frequency resolution by adding “zero information” — only by analyzing more data.

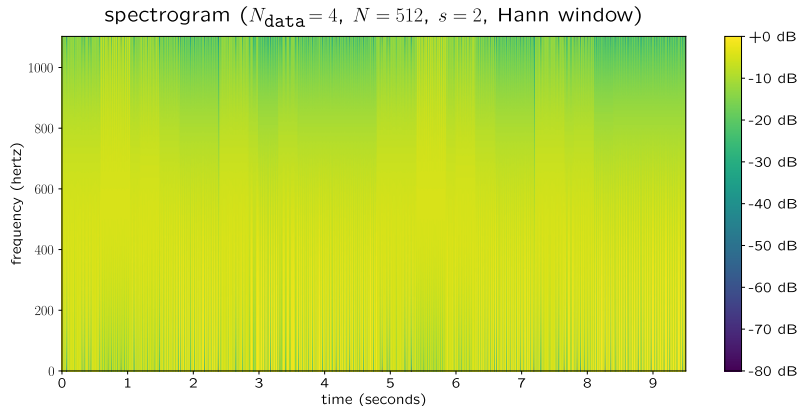
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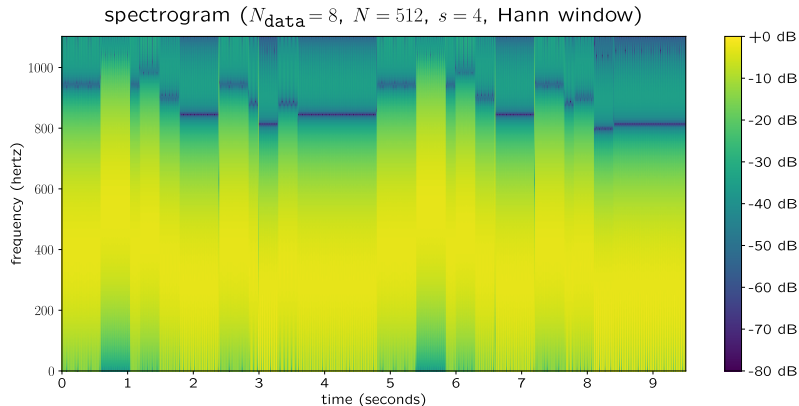
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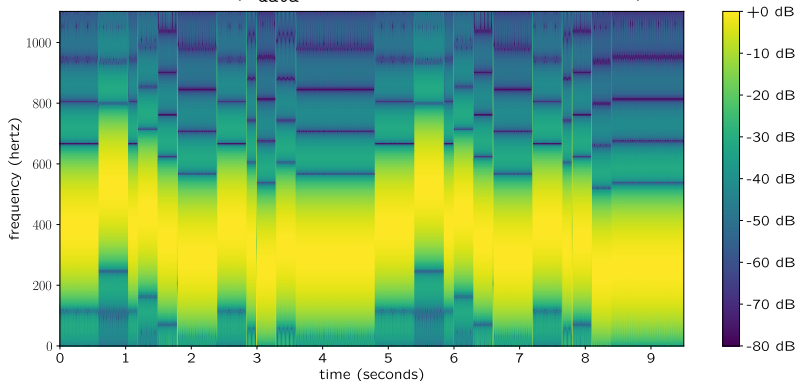


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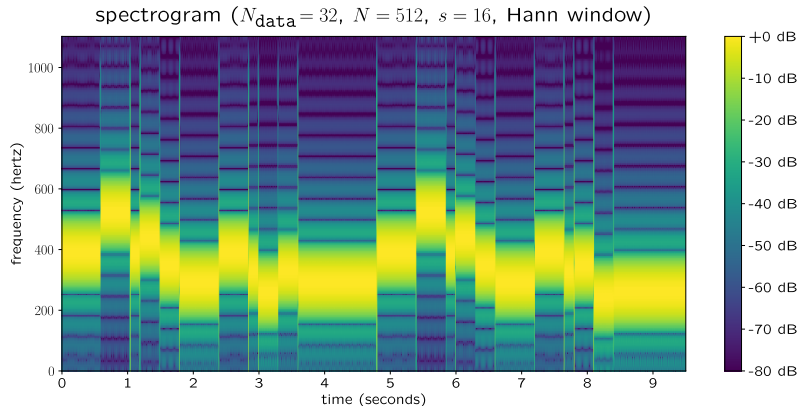
spectrogram ($N_{\text{data}} = 16$, $N = 512$, $s = 8$, Hann window)



Decibels: $x_{\text{dB}} = 10 \log_{10}(x)$

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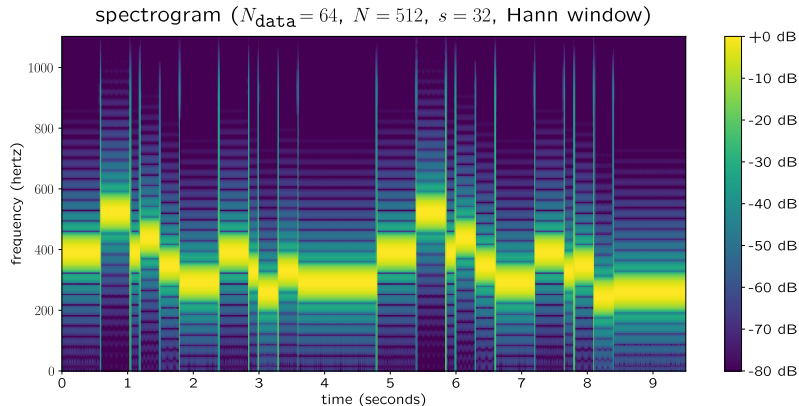
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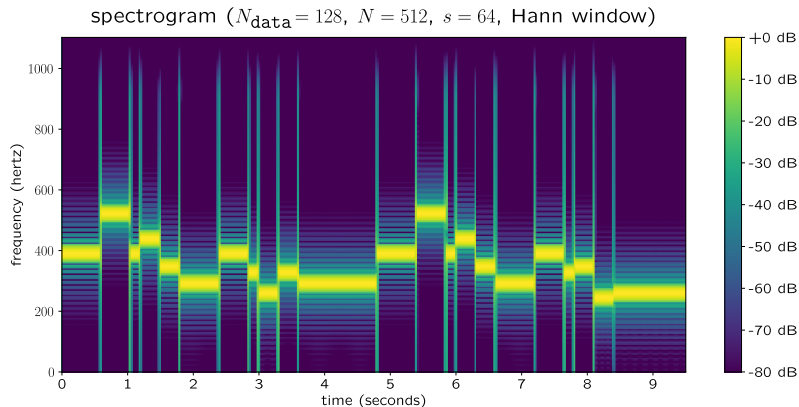
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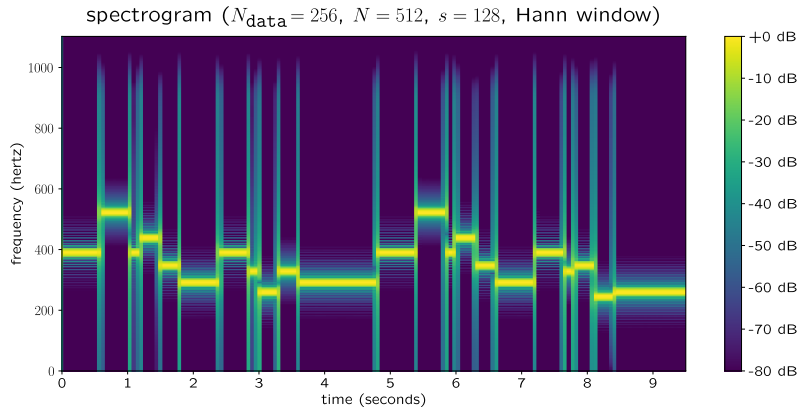
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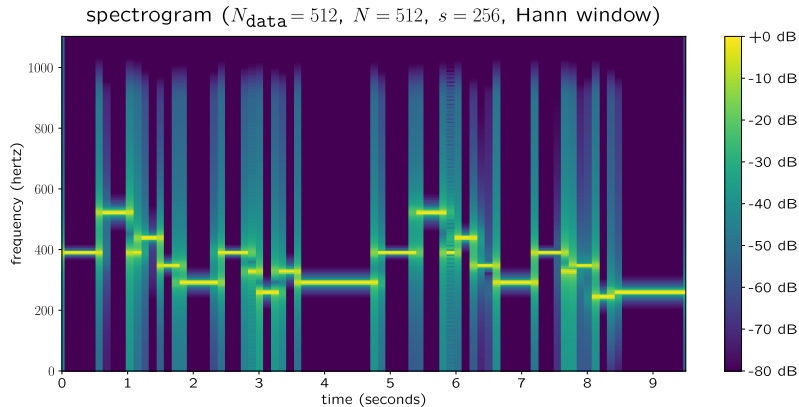
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Short-Time Fourier Transform

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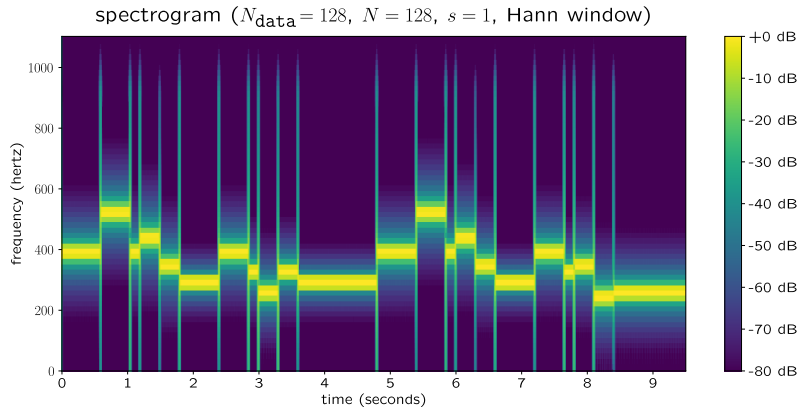
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- DFT size: N
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Short-Time Fourier Transform

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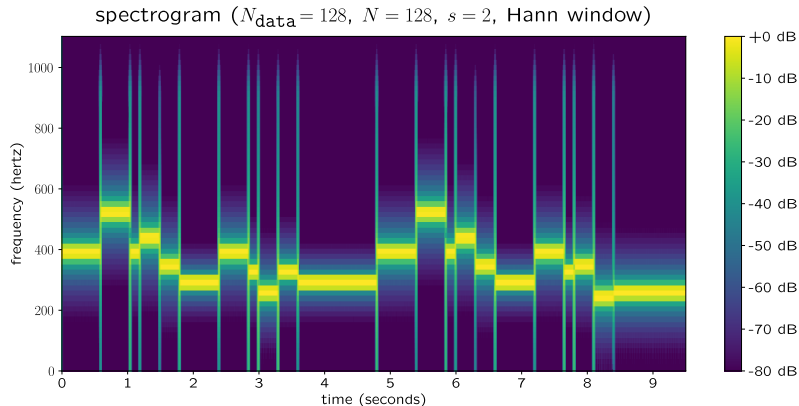
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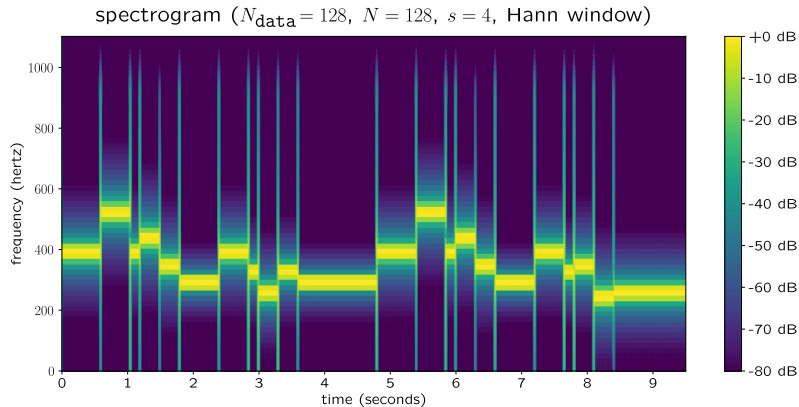
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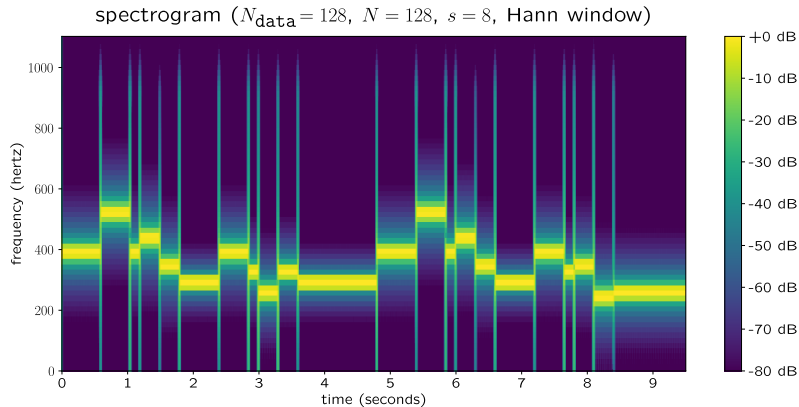
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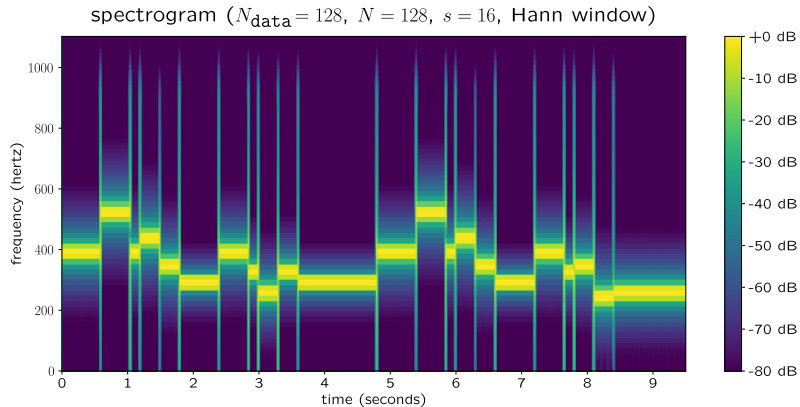
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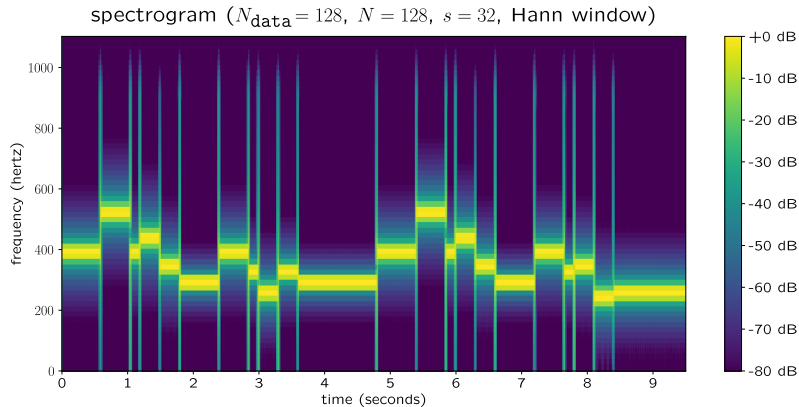
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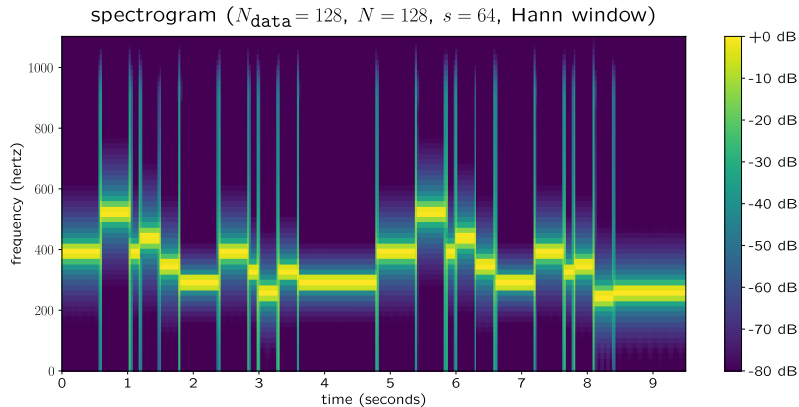
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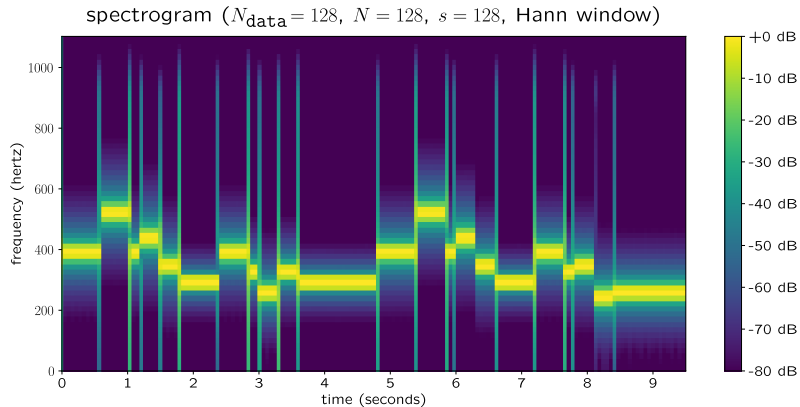
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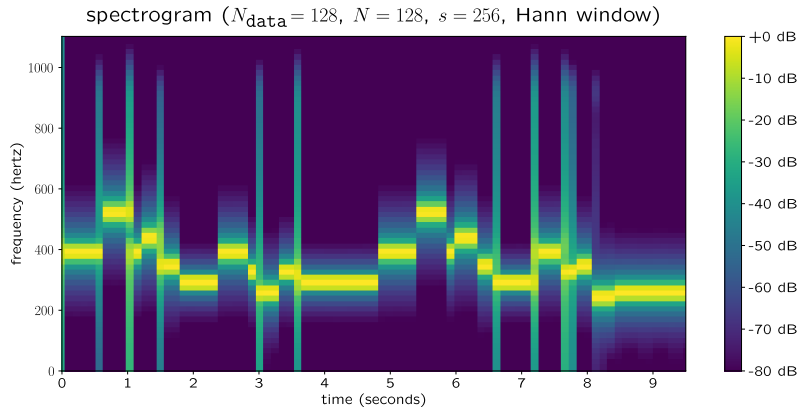
STFT: Step Size



Decibels: $x_{\text{dB}} = 10 \log_{10}(x)$

e.g., $-10 \text{ dB} \iff x = 10^{-1}$ and $-20 \text{ dB} \iff x = 10^{-2}$

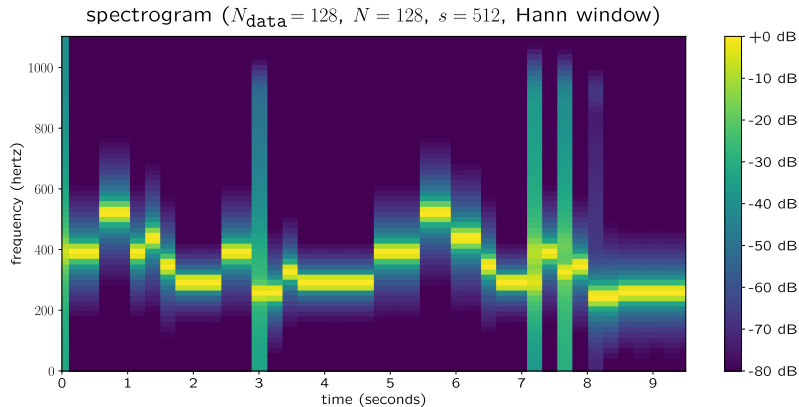
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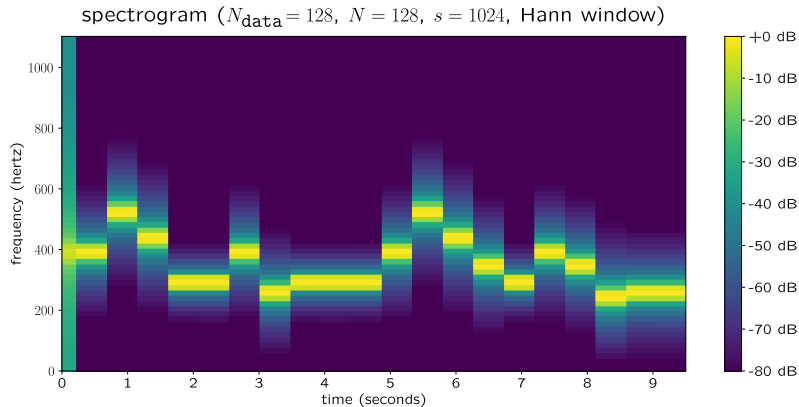
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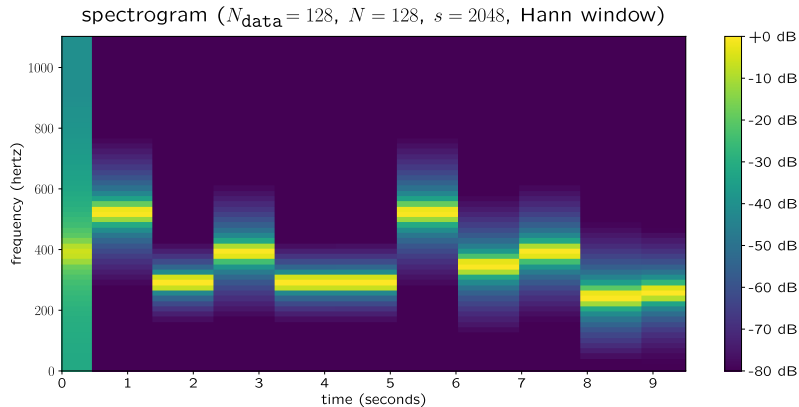
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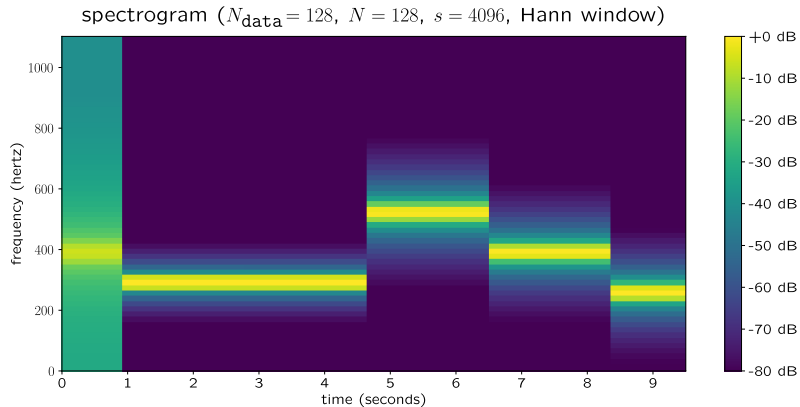
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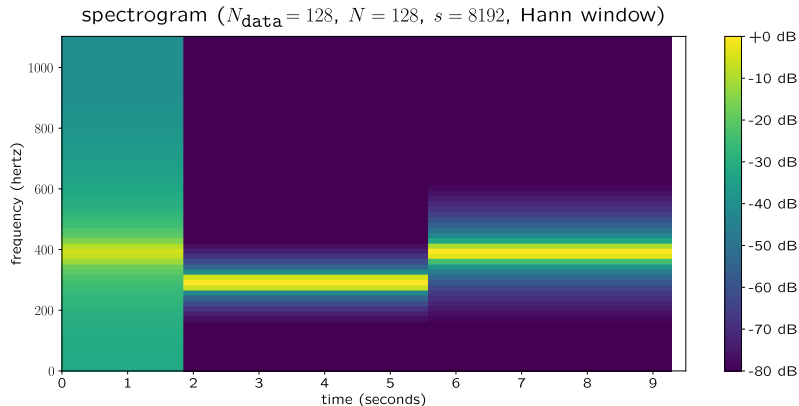
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Short-Time Fourier Transform

Parameters:

- type of window: $w[n]$
- DFT size: N
- step size: s

Lessons Learned

A **short-time Fourier transform (STFT)** is a sequence of windowed Fourier transforms. A **spectrogram** displays the (squared) magnitude of an STFT.

Windows: Consider the main lobe and sidelobes.

Time Resolution vs. Frequency Resolution

- time resolution: short $w[n]$ $\iff$ wide $W(\Omega)$
- frequency resolution: long $w[n]$ $\iff$ narrow $W(\Omega)$

Analysis:
$$X[k, m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n + ms] w[n] e^{-jk \frac{2\pi}{N} n}$$

Spectrogram: $|X[k, m]|^2$ (magnitude² of STFT)

Question of the Day

Describe what each STFT parameter means.

- type of window: $w[n]$
- DFT size: N
- length of data: N_{data}
- step size: s