6.300: Signal Processing

Discrete Fourier Transform

Analysis: $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$

Synthesis: $x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}$

Relating the DFT to DTFS and DTFT:

- DTFS of periodically-extended $x_w[n] = x[n]w[n]$
- Sampled ($\Omega \to 2\pi k/N$), scaled ($1/N$) DTFT of $x_w[n]$

Frequency resolution: $\Delta f = f_s/N$ and $\Delta\Omega = 2\pi/N$

Agenda for Recitation

- Fourier representations (so far)
- Spectral analysis with the DFT

What questions do you have from lecture?

Agenda for Recitation

- Fourier representations (so far)
- Spectral analysis with the DFT

Fourier Representations

	type of time-domain signal			domain	period
CTFS	CT	period T	infinite length	integer k	none
DTFS	DT	period N	infinite length	integer k	N
CTFT	CT	aperiodic	infinite length	real ω	none
DTFT	DT	aperiodic	infinite length	real Ω	2π
DFT	DT	aperiodic	length N	integer k	N

The **discrete Fourier transform (DFT)** is a discrete-time, discrete-frequency Fourier transform.

- for aperiodic discrete-time (n) signals $x[n]$
- yields discrete-frequency (k) representation $X[k]$
- finite length (N) in both time and frequency
- closely related to DTFS and DTFT

Discrete-Time Fourier Representations

Another Fourier transform? Why?

DFT: Periodic Extension

Relation to DTFS: The DFT yields the DTFS coefficients for a periodically-extended $x_w[n] = x[n]w[n]$.

DTFS: $X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n}$ **DFT:** $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$

Let $N = 32$. How many of the following signals have "simple" DFTs (i.e., mostly zeros)?

- $x_1[n] = \cos\left(\frac{2\pi}{32}n\right)$
- $x_2[n] = \cos\left(\frac{2\pi}{64}n\right)$
- $x_3[n] = \cos\left(\frac{2\pi}{16}n - \frac{\pi}{2}\right)$
- $x_4[n] = \cos\left(\frac{2\pi}{4}n\right)\cos\left(\frac{2\pi}{8}n\right)$
- $x_5[n] = \cos\left(\frac{2\pi}{32}n\right) + j\sin\left(\frac{2\pi}{32}n\right)$

DFT: Sampling the DTFT

Relation to DTFT: Multiplication by a window function $w[n]$ in time corresponds to convolution with the Fourier transform of the window function, $W(\Omega)$, in frequency.

$$x_w[n] = x[n]w[n] \iff X_w(\Omega) = \frac{1}{2\pi}(X * W)(\Omega)$$

The DFT is a sampled ($\Omega \to \frac{2\pi}{N}k$), scaled ($1/N$) DTFT.

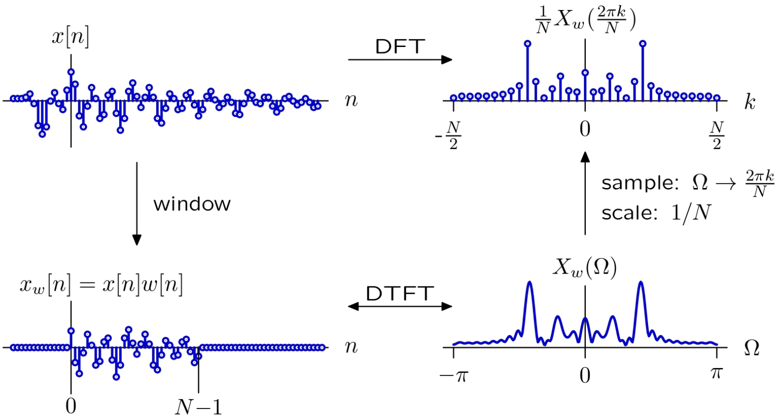
$$X[k] = \frac{1}{N}X_w(\frac{2\pi}{N}k)$$

Notice that the distance between adjacent DFT frequency samples (or "bins") is $\Delta\Omega = 2\pi/N$ radians per sample. Accordingly, as N increases, $\Delta\Omega$ decreases, and this yields finer frequency resolution.[1]

[1] In general, the length of the DFT analysis window (N) can exceed the length of the data. Ignore this case ("zero-padding") for now.

Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



Graphic: Professor Denny Freeman (freeman@mit.edu)

Agenda for Recitation

- Fourier representations (so far)
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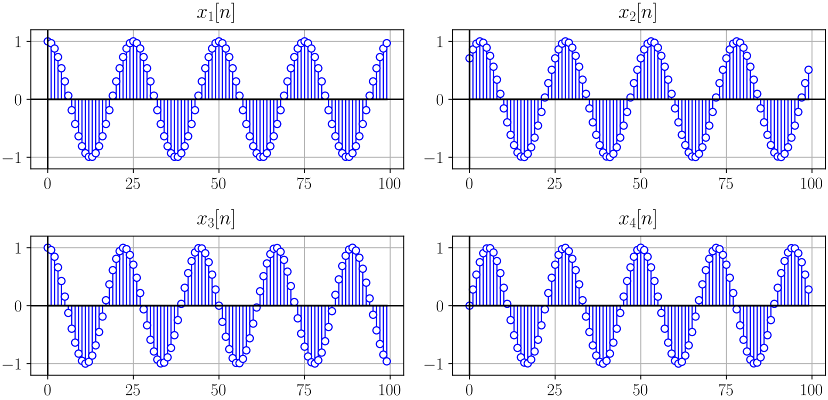
Spectral Analysis with the DFT

Consider the four discrete-time signals below.

$$
\begin{aligned}
x_1[n] &= \cos\left(\tfrac{8\pi}{100}n\right) \\
x_2[n] &= \cos\left(\tfrac{8\pi}{100}n - \tfrac{\pi}{4}\right) \\
x_3[n] &= \cos\left(\tfrac{9\pi}{100}n\right) \\
x_4[n] &= \cos\left(\tfrac{9\pi}{100}n - \tfrac{\pi}{2}\right)
\end{aligned}
$$

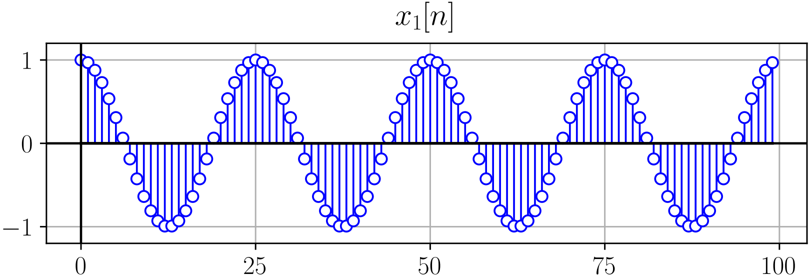
Suppose that the sampling rate is `fs` $= 44100$ samples per second. How would you write a Python program to generate a second's worth of samples for each of the signals above?

Spectral Analysis with the DFT



Spectral Analysis with the DFT

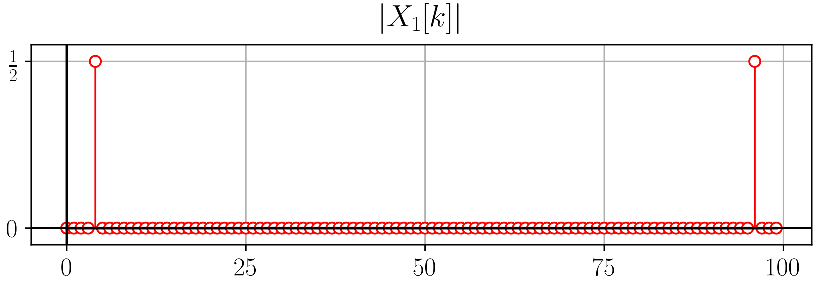
The first $N = 100$ samples of $x_1[n]$ are plotted below.



Recall that the sampling rate is $\texttt{fs} = 44100$ samples per second. What continuous-time cyclical frequency f_0 (cycles per second) does discrete-time angular frequency $\Omega_0 = \frac{8\pi}{100}$ (radians per sample) correspond to?

Spectral Analysis with the DFT

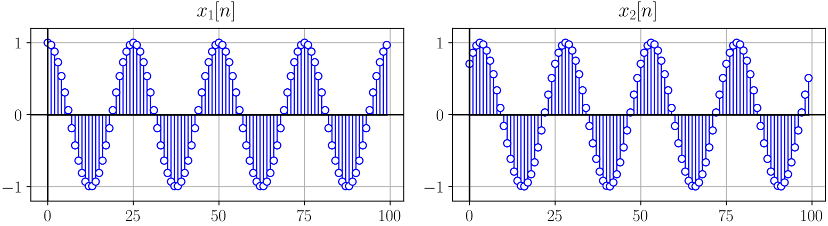
Shown below is $|X_1[k]|$, the magnitude of the DFT of the first $N = 100$ samples of $x_1[n]$.



For which values of k is $|X_1[k]|$ non-zero? Explain.

Spectral Analysis with the DFT

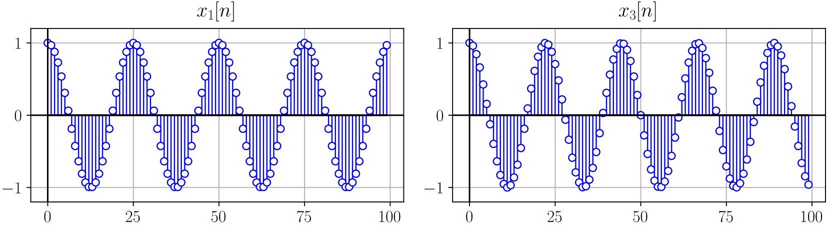
The first $N = 100$ samples of $x_1[n]$ and $x_2[n]$ are plotted below.



How will $|X_1[k]|$ and $|X_2[k]|$ differ?

Spectral Analysis with the DFT

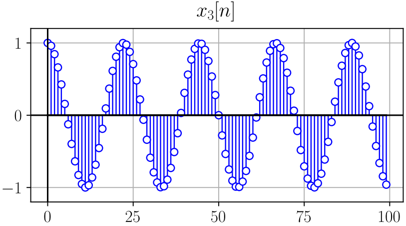
The first $N = 100$ samples of $x_1[n]$ and $x_3[n]$ are plotted below.



How will $|X_1[k]|$ and $|X_3[k]|$ differ?

Spectral Analysis with the DFT

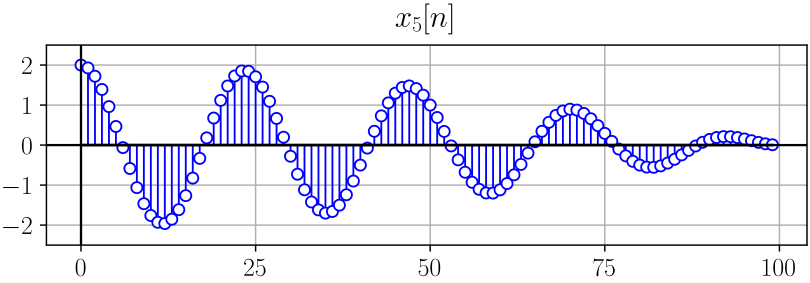
The first $N = 100$ samples of $x_3[n]$ and $x_4[n]$ are plotted below.



How will $|X_3[k]|$ and $|X_4[k]|$ differ?

Spectral Analysis with the DFT

Consider $x_5[n] = \cos\left(\frac{8\pi}{100}n\right) + \cos\left(\frac{9\pi}{100}n\right)$. The first $N = 100$ samples are plotted below.



What is the minimum analysis window length N that's needed to resolve $\Omega_1 = \frac{8\pi}{100}$ from $\Omega_2 = \frac{9\pi}{100}$?

Spectral Analysis with the DFT

Frequency resolution is proportional to $1/N$. Suppose that we append a bunch of zeros to the end of our finite-length signal to make it longer. This increases N. Does our frequency resolution improve? Explain.

Lessons Learned

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