

6.300: Signal Processing

Frequency Response and Filtering

An LTI system's **frequency response** is the Fourier transform of the system's unit-sample response.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} \iff h[n] = \frac{1}{2\pi} \int_{2\pi} H(\Omega) e^{j\Omega n} d\Omega$$

Filtering: Convolution (*) in the time domain corresponds to multiplication (×) in the frequency domain.

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = (h * x)[n]$$

$$X(\Omega) \rightarrow \boxed{\text{LTI}} \rightarrow Y(\Omega) = H(\Omega)X(\Omega)$$

Agenda for Recitation

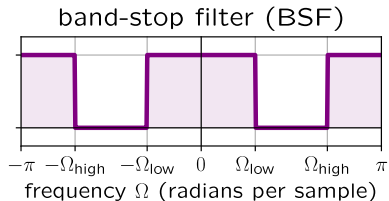
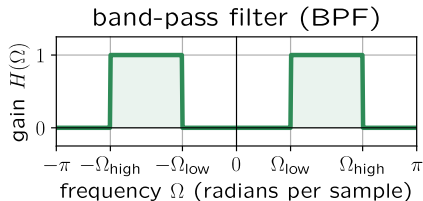
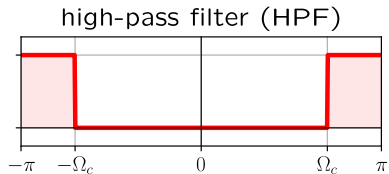
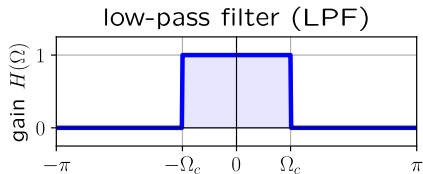
- Filters: Low-pass, high-pass, band-pass, and band-stop
- Connecting systems in series: Cascaded system
- Analog and digital systems; filter design (optional)
- Fourier transforms in probability theory (optional)

What questions do you have from lecture?

Agenda for Recitation

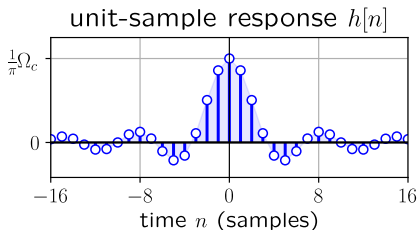
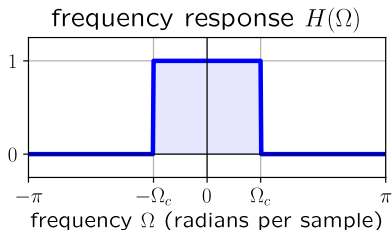
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Filters



- **LPF**: pass low frequencies, reject high frequencies
- **HPF**: pass high frequencies, reject low frequencies
- **BPF**: pass specified band of frequencies, reject others
- **BSF**: reject specified band of frequencies, pass others

Filters



Suppose that we low-pass filter a signal.

$$X(\Omega) \rightarrow \boxed{\text{LPF}} \rightarrow Y(\Omega) = H(\Omega)X(\Omega)$$

$$\text{or } x[n] \rightarrow \boxed{\text{LPF}} \rightarrow y[n] = (h * x)[n]$$

Multiplication ($\times$) by $H(\Omega)$ in the frequency domain is equivalent to convolution ($*$) with $h[n]$ in the time domain. Determine a closed-form expression for $h[n]$.

Filters

Multiplication ($\times$) by $H(\Omega)$ in the frequency domain is equivalent to convolution ($*$) with $h[n]$ in the time domain. Determine a closed-form expression for $h[n]$.

$$h[n] = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} H(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega n} d\Omega = \frac{\sin(\Omega_c n)}{\pi n}$$

This takes the form of a **sinc** (“sinus cardinalis” or “sine cardinal”) function.

$$\text{**sinc**: } \frac{\sin(x)}{x} \qquad \text{**normalized sinc**: } \frac{\sin(\pi x)}{\pi x}$$

Writing $\text{sinc}(x)$ is confusing because it's not clear whether it refers to the normalized sinc or not. Don't do it here.

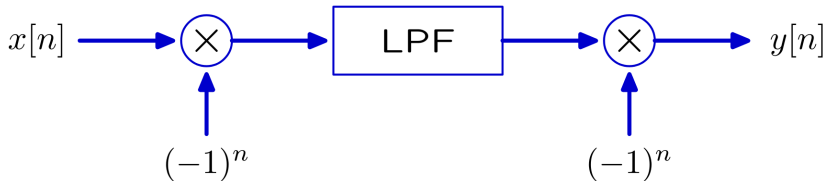
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Cascaded System

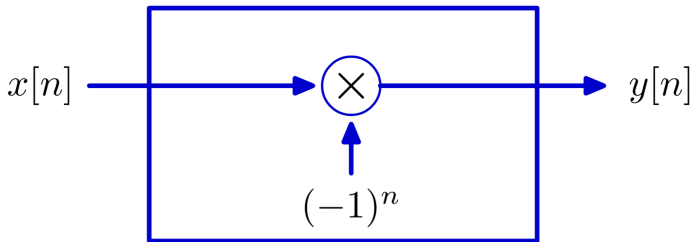


Consider the cascaded system shown above.

How many statements below are true?

- The cascaded system is **linear**.
- The cascaded system is **time-invariant**.

Cascaded System: Multiplier



Let's break the problem down into sub-problems.
Consider the multiplier sub-system shown above.

How many statements below are true?

- The multiplier is **linear**.
- The multiplier is **time-invariant**.

Cascaded System: Multiplier

Is the multiplier linear? Yes.

Assume that $y_i[n]$ is the response to $x_i[n]$.

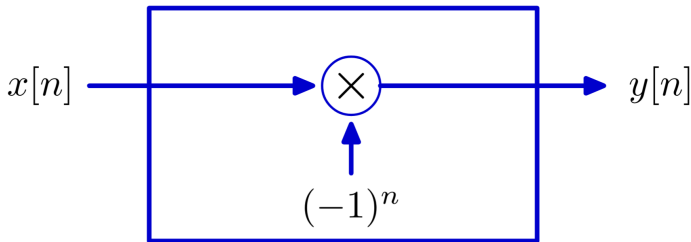
$$\begin{aligned}y[n] &= (-1)^n x[n] \\&= (-1)^n (c_1 x_1[n] + c_2 x_2[n]) \\&= c_1 (-1)^n x_1[n] + c_2 (-1)^n x_2[n] \\&= c_1 y_1[n] + c_2 y_2[n]\end{aligned}$$

Is the multiplier time-invariant? No.

Assume that $y'[n]$ is the response to $x'[n]$.

$$\begin{aligned}y[n] &= (-1)^n x[n] \\&= (-1)^n x'[n - n_0] \\&= (-1)^{n_0} (-1)^{n-n_0} x'[n - n_0] \\&= (-1)^{n_0} y'[n - n_0]\end{aligned}$$

Cascaded System: Multiplier



Let's break the problem down into sub-problems.
Consider the multiplier sub-system shown above.

How many statements below are true? **1**

- The multiplier is **linear**.
- The multiplier is **time-invariant**.

yes
no

Cascaded System: Low-Pass Filter



Let's break the problem down into sub-problems.
Consider the low-pass filter sub-system shown above.

How many statements below are true?

- The low-pass filter is **linear**.
- The low-pass filter is **time-invariant**.

Cascaded System: Low-Pass Filter

Is the low-pass filter linear? Yes.

Assume that $Y_i(\Omega)$ is the response to $X_i(\Omega)$.

$$\begin{aligned} Y(\Omega) &= H(\Omega)X(\Omega) \\ &= H(\Omega)(c_1X_1(\Omega) + c_2X_2(\Omega)) \\ &= c_1H(\Omega)X_1(\Omega) + c_2H(\Omega)X_2(\Omega) \\ &= c_1Y_1(\Omega) + c_2Y_2(\Omega) \end{aligned}$$

Is the low-pass filter time-invariant? Yes.

Assume that $Y'(\Omega)$ is the response to $X'(\Omega)$.

$$\begin{aligned} Y(\Omega) &= H(\Omega)X(\Omega) \\ &= H(\Omega)e^{-j\Omega n_0}X'(\Omega) \\ &= e^{-j\Omega n_0}H(\Omega)X'(\Omega) \\ &= e^{-j\Omega n_0}Y'(\Omega) \end{aligned}$$

Cascaded System: Low-Pass Filter

We could perform the analysis in the time domain as well.

Is the low-pass filter linear? Yes.

Assume that $y_i[n]$ is the response to $x_i[n]$.

$$\begin{aligned}y[n] &= (x * h)[n] \\&= \sum_m x[m]h[n - m] \\&= \sum_m (c_1 x_1[m] + c_2 x_2[m])h[n - m] \\&= c_1 \sum_m x_1[m]h[n - m] + c_2 \sum_m x_2[m]h[n - m] \\&= c_1 y_1[n] + c_2 y_2[n]\end{aligned}$$

Cascaded System: Low-Pass Filter

We could perform the analysis in the time domain as well.

Is the low-pass filter time-invariant? Yes.

Assume that $y'[n]$ is the response to $x'[n]$.

$$\begin{aligned}y[n] &= (x * h)[n] \\&= \sum_m x[m] h[n - m] \\&= \sum_m x'[m - n_0] h[n - m] \\&= \sum_{\ell} x'[\ell] h[n - (n_0 + \ell)] \\&= \sum_{\ell} x'[\ell] h[(n - n_0) - \ell] \\&= y'[n - n_0]\end{aligned}$$

Cascaded System: Low-Pass Filter

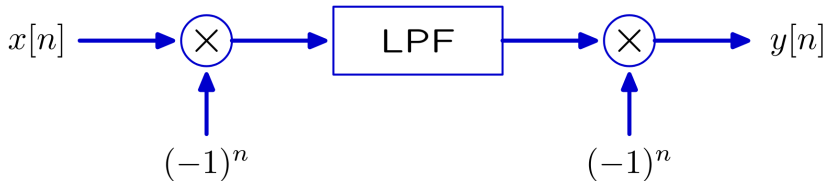


Let's break the problem down into sub-problems.
Consider the low-pass filter sub-system shown above.

How many statements below are true? **2**

- The low-pass filter is **linear**. **yes**
- The low-pass filter is **time-invariant**. **yes**

Cascaded System

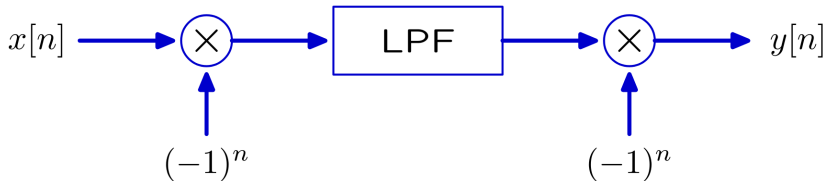


Consider the cascaded system shown above.

How many statements below are true?

- The cascaded system is **linear**.
- The cascaded system is **time-invariant**.

Cascaded System



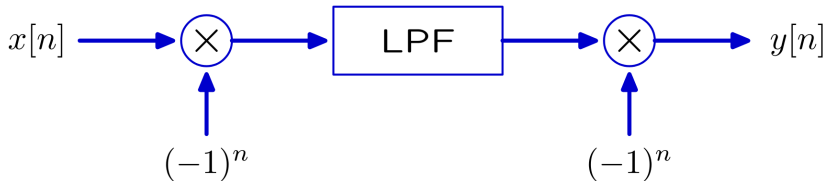
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How many statements below are true?

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- The cascaded system is **time-invariant**.

The composition of linear systems is always linear.
However, the composition of time-varying systems may be time-invariant! e.g., $(-1)^n \times (-1)^n = (-1)^{2n} = 1$

Cascaded System



Consider the cascaded system shown above.

How many statements below are true? **2**

- The cascaded system is **linear**. **yes**
- The cascaded system is **time-invariant**. **yes**

Is the cascaded system best described as a low-pass filter (LPF) or high-pass filter (HPF)?

Cascaded System: High-Pass Filter?

The time-domain relation between input and output is

$$\begin{aligned}y[n] &= (-1)^n \sum_m (-1)^m x[m] h[n - m] \\&= \sum_m (-1)^{n+m} x[m] h[n - m] \\&= \sum_m (-1)^{n-m} x[m] h[n - m] \\&= \sum_m x[m] (-1)^{n-m} h[n - m] \\&= \sum_m x[m] h'[n - m]\end{aligned}$$

So, is $h'[n] \triangleq (-1)^n h[n]$ a high-pass filter?

Cascaded System: High-Pass Filter?

Compute the frequency response.

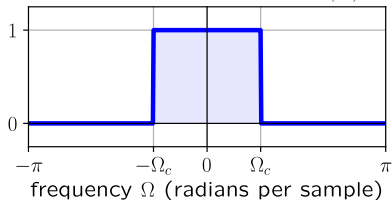
$$\begin{aligned}H'(\Omega) &= \sum_n h'[n] e^{-j\Omega n} \\&= \sum_n (-1)^n h[n] e^{-j\Omega n} \\&= \sum_n e^{j\pi n} h[n] e^{-j\Omega n} \\&= \sum_n h[n] e^{-j(\Omega - \pi)n} \\&= H(\Omega - \pi)\end{aligned}$$

This is an example of the **frequency-shifting property**.

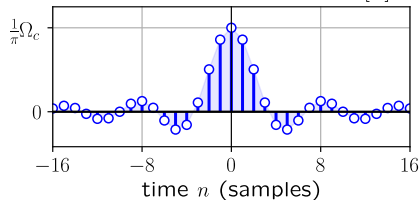
$$e^{j\Omega_0 n} h[n] \iff H(\Omega - \Omega_0)$$

Cascaded System: High-Pass Filter

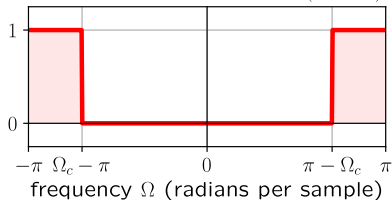
frequency response $H(\Omega)$



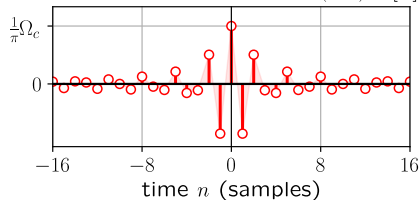
unit-sample response $h[n]$



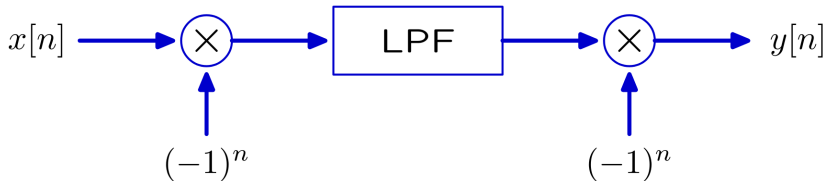
frequency response $H(\Omega - \pi)$



unit-sample response $(-1)^n h[n]$



Cascaded System



Consider the cascaded system shown above.

How many statements below are true? **2**

- The cascaded system is **linear**. **yes**
- The cascaded system is **time-invariant**. **yes**

Is the cascaded system best described as a low-pass filter (LPF) or high-pass filter (HPF)? **HPF**

Agenda for Recitation

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The material presented after this point is **not necessary** for success in this subject; it is provided only for your enrichment. Perhaps you'll find it interesting.

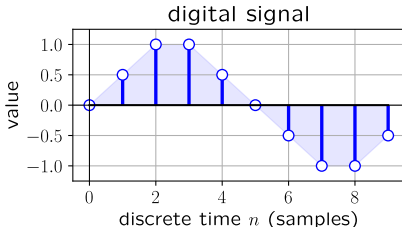
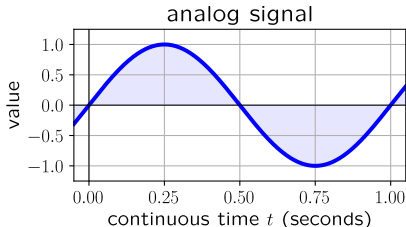
Analog and Digital Systems

Analog means “continuous in both the independent and dependent variables.” **Digital** means “discrete in both the independent and dependent variables.”

We often think of time (t , n) or frequency (ω , Ω) as the independent variables.

Digitization (analog $\rightarrow$ digital) entails two steps.

- **sampling** (e.g., continuous time $t \rightarrow$ discrete time n)
- **quantization** (storing values with finite register length)



Analog and Digital Systems

Analog Filters

- electrical (resistor, capacitor, inductor)
- mechanical (mass, spring, damper)
- acoustical (mass, compliance, resistance)
- biological (e.g., phosphorylation cycle¹)

Digital Filters

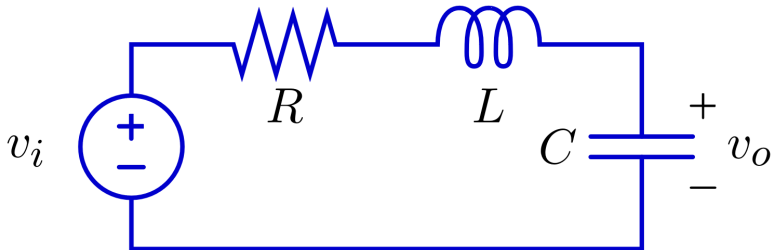
- array(s) of numbers
 - coefficients ($b_0, b_1, \dots, a_0, a_1, \dots$)
 - zeros, poles, and gain ($\xi_1, \xi_2, \dots, p_1, p_2, \dots, K$)
 - state-space representation (A, B, C, D)

¹*Biomolecular Feedback Systems* by Del Vecchio and Murray

Analog and Digital Systems

Electrical System: Series RLC Circuit

- input voltage $v_i = v_i(t)$ volts (V)
- output voltage $v_o = v_o(t)$ volts (V)
- resistor with resistance R ohms (Ω)
- capacitor with capacitance C farads (F)
- inductor with inductance L henries (H)



Analog and Digital Systems

Electrical System: Series RLC Circuit

The relation between v_i and v_o takes the form of a linear, constant-coefficient differential equation.

$$LC \frac{d^2 v_o(t)}{dt^2} + RC \frac{dv_o(t)}{dt} + v_o(t) = v_i(t)$$

The frequency response may be written as

$$H(\omega) \triangleq \frac{V_o(\omega)}{V_i(\omega)} = \frac{\omega_0^2}{\omega_0^2 + \frac{1}{\tau}j\omega - \omega^2}$$

where

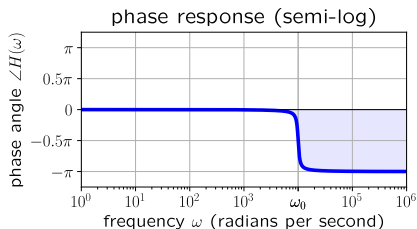
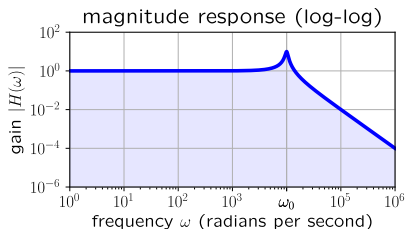
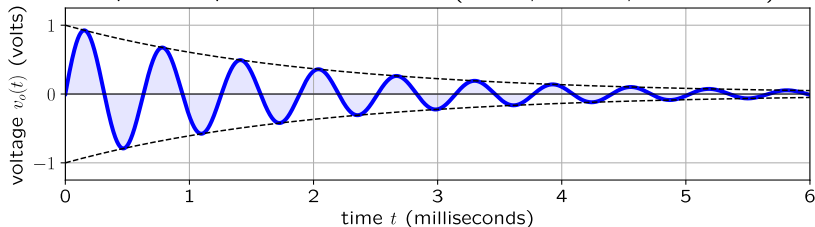
$$\omega_0 \triangleq \sqrt{\frac{1}{LC}} \text{ and } \tau \triangleq \frac{L}{R}$$

denote the **natural frequency** and **time constant**, respectively.

Analog and Digital Systems

Electrical System: Series RLC Circuit

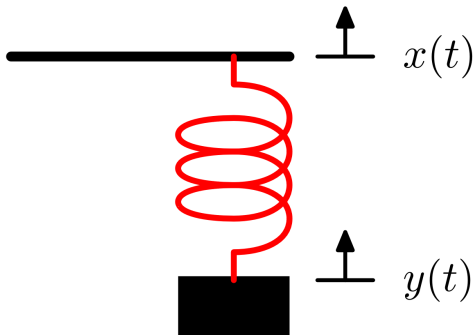
impulse response of series RLC ($R = 10$, $L = 0.01$, $C = 0.000001$)



Analog and Digital Systems

Mechanical System: Mass on Spring

- input position $x(t)$ meters (m)
- output position $y(t)$ meters (m)
- mass M kilograms (kg)
- spring constant K newtons per meter (N/m)



Analog and Digital Systems

Mechanical System: Mass on Spring

The relation between x and y takes the form of a linear, constant-coefficient differential equation.

$$M \frac{d^2 y(t)}{dt^2} = K(x(t) - y(t))$$

The frequency response may be written as

$$H(\omega) \triangleq \frac{Y(\omega)}{X(\omega)} = \frac{\omega_0^2}{\omega_0^2 - \omega^2}$$

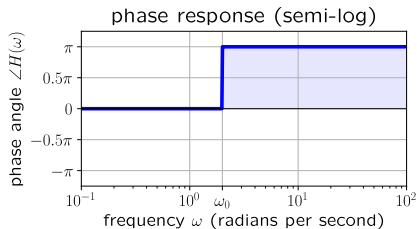
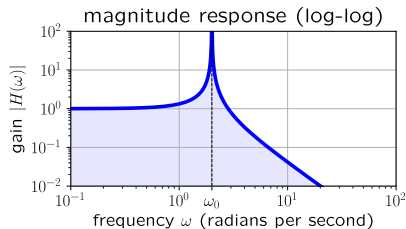
where

$$\omega_0 \triangleq \sqrt{\frac{K}{M}}$$

denotes the **natural frequency**.

Analog and Digital Systems

Mechanical System: Mass on Spring



Resonance occurs at the natural frequency ω_0 . Pure resonance may only be attained without any damping.

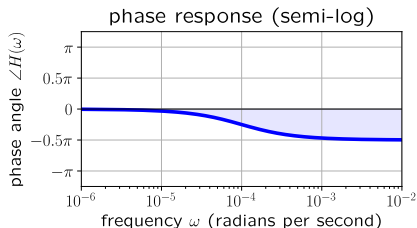
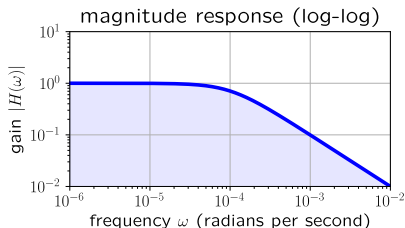
Electrical-mechanical analogies: For linear **electrical systems**, the “basic building blocks” (i.e., lumped elements) are resistors, inductors, and capacitors. For linear **mechanical systems**, the “basic building blocks” (i.e., lumped elements) are springs, masses, and dampers.

Analog and Digital Systems

Biological System: Phosphorylation Cycle²

- kinase $x(t)$
- phosphorylated substrate $y(t)$
- production rate β
- decay rate γ

$$\frac{dy(t)}{dt} = \beta x(t) - \gamma y(t)$$



Analog and Digital Systems

Per Professor Al Oppenheim, pre-1960s signal processing was essentially **analog**. *People built signal processing systems with . . . resistors, inductors, capacitors, op amps, and things like that.*³

J. W. Cooley and J. Tukey's re-discovery of the **FFT** circa 1965 marked the advent of **digital signal processing (DSP)** as we now know it — processing signals with digital computers.

Dr. Tom Barnwell (of Georgia Tech) once joked,

*DSP is a discipline that allows us to replace a simple resistor and capacitor with two anti-aliasing filters, an [analog-to-digital] and [digital-to-analog] converter, and a general purpose computer . . . as long as the signal we are interested in doesn't vary too quickly.*⁴

³https://ethw.org/Oral-History:Alan_Oppenheim

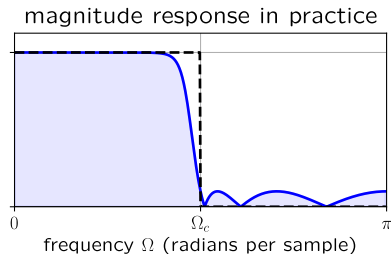
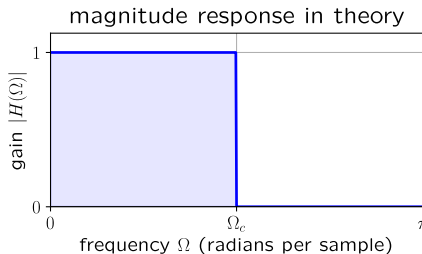
⁴<https://www.eecs.mit.edu/signal-processing-how-did-we-get-to-were-were-going/>

Digital Filter Design

“Ideal filters” simplify analyses. They have

- perfectly flat **passbands**,⁵
- perfectly flat **stopbands**,⁶ and
- arbitrarily narrow **transition bands**,⁷

but they are not realizable in practice.



⁵range of frequencies you want to keep

⁶range of frequencies you want to reject

⁷region between passband and stopband

Digital Filter Design

Filter design: Design a filter to satisfy specifications. Some compromises or trade-offs may need to be made.

#1. Define the desired response in time or frequency.

- e.g., **passband** and **stopband** frequencies

#2. Define **specifications**. (You must allow some “wiggle room” for a practical implementation.)

- maximum and minimum **passband gain**
- maximum and minimum **stopband gain**
- **filter order** (i.e., maximum length of $h[n]$)

#3. Choose a design method.

- **ad hoc** (e.g., intuition, experimentation, ...)
- **optimal** (Solve an optimization problem.)⁸

⁸**Optimal** means that you solved some optimization problem. It doesn't mean that the filter you designed is any good, though.

Digital Filter Design⁹

Infinite Impulse Response (IIR)

- Difference equation: $\sum_k a_k y[n - k] = \sum_k b_k x[n - k]$
- Can be unstable: $\sum_n |(h * x)[n]| \rightarrow \infty$ even if $\sum_n |x[n]| < \infty$
- Can be difficult to control phase response $\angle H(\Omega)$
- Generally fewer adds, multiplies required than FIR filter
- Significant history in analog filter design

Finite Impulse Response (FIR)

- Difference equation: $y[n] = \sum_k b_k x[n - k]$
- Always stable: If $\sum_n |x[n]| < \infty$, then $\sum_n |(h * x)[n]| < \infty$
- Linear phase $\angle H(\Omega) = \alpha_0 \Omega + \beta_0$ usually achievable
- Generally more adds, multiplies required than IIR filter
- Potential for polyphase filter bank implementation
- Can always be made causal: $h[n] = 0$ for $n < 0$
- Some relevant history in antenna array design

⁹from 6.7000: Discrete-Time Signal Processing (2024)

Digital Filter Design

FIR Design Methods

- Window method: Truncate unit-sample response.
- Frequency sampling: Sample $H(\Omega)$; compute $\tilde{h}[n]$.
- Least squares: Minimize squared error.
- Constrained least squares (e.g., peak gain constraints)
- Minimax: Minimize the maximum deviation.
 - e.g., via Parks-McClellan/Remez exchange algorithm

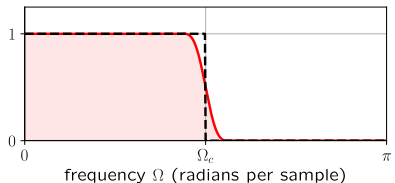
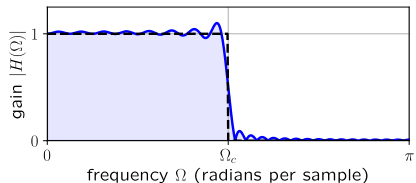
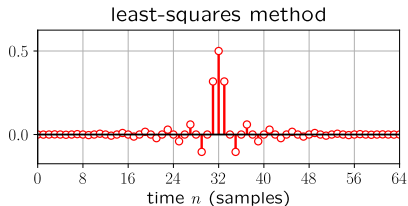
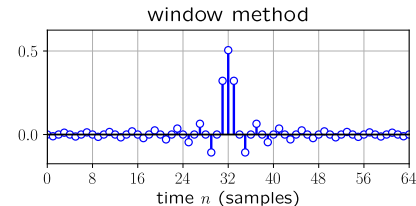
IIR Design Methods

- Impulse invariance: Sample $h(t)$ to get $h[n] \triangleq h(n\Delta)$.
- Bilinear transform: Map $\omega \in (-\infty, \infty)$ to $\Omega \in [-\pi, \pi]$.
 - Butterworth
 - Chebyshev (Type I and Type II)
 - Elliptic (Cauer)

... and more!

Digital Filter Design

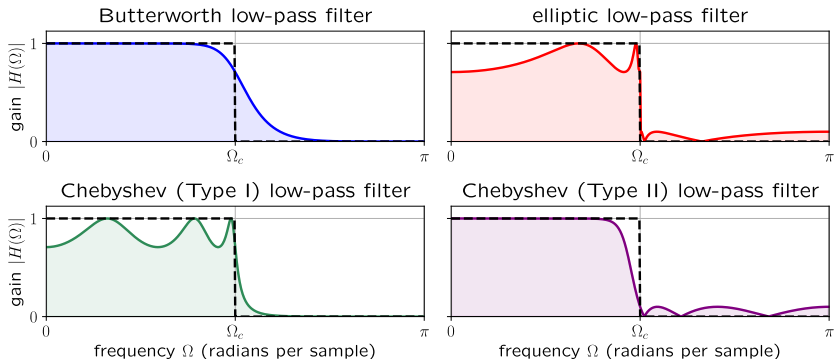
Examples of FIR Digital Filters



- cut-off frequency: 0.5π radians per sample
- transition width: 0.1π radians per sample
- filter order (number of non-zero taps): 65

Digital Filter Design

Examples of IIR Digital Filters



- cut-off frequency: 0.5π radians per sample
- maximum passband ripple: 3 dB
- minimum stopband attenuation: 20 dB
- filter order: 6

Agenda for Recitation

- Filters: Low-pass, high-pass, band-pass, and band-stop
- Connecting systems in series: Cascaded system
- Analog and digital systems; filter design (optional)
- Fourier transforms in probability theory (optional)

The material presented after this point is **not necessary** for success in this subject; it is provided only for your enrichment. Perhaps you'll find it interesting.

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Fourier Transforms in Probability

A **probability density function (PDF)** $p_X(t)$ defines the probability distribution of a **random variable** X .

Take on faith the fact that the random variable $Z = X + Y$ has probability density function $p_Z(t) = (p_X * p_Y)(t)$.

The Fourier transform of the probability density function is called the **characteristic function**.

$$\text{PDF } p_X(t) \iff \text{characteristic function } \varphi_X(\omega)$$

While the moment generating function (MGF) may not always exist, the characteristic function always exists.

Convolution of PDFs is equivalent to multiplication of the corresponding characteristic functions.

$$p_Z(t) = (p_X * p_Y)(t) \iff \varphi_Z(\omega) = \varphi_X(\omega)\varphi_Y(\omega)$$

And We're Back

We now return to our regularly-scheduled programming.

Lessons Learned

We may analyze and design LTI systems in both the time (n) and frequency (Ω) domains. It's natural to describe many systems in terms of a frequency response $H(\Omega)$.

An LTI system's **frequency response** is the Fourier transform of the system's unit-sample response.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} \iff h[n] = \frac{1}{2\pi} \int_{2\pi} H(\Omega) e^{j\Omega n} d\Omega$$

Filtering: Convolution ($*$) in the time domain corresponds to multiplication ($\times$) in the frequency domain.

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = (h * x)[n]$$

$$X(\Omega) \rightarrow \boxed{\text{LTI}} \rightarrow Y(\Omega) = H(\Omega)X(\Omega)$$

Question of the Day

The discrete-time analogue of taking a derivative is taking a **finite difference**, like

$$y[n] \triangleq x[n+1] - x[n].$$

Assume that this LTI system starts at rest. What is the unit-sample response $h[n]$? What is the frequency response $H(\Omega)$? Is this LTI system best described as a low-pass filter or as a high-pass filter?