

6.300: Signal Processing

Systems: Linearity and Time-Invariance

Additivity: $x_1[n] + x_2[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y_1[n] + y_2[n]$

Homogeneity: $cx[n] \rightarrow \boxed{\mathcal{H}} \rightarrow cy[n]$ for any constant c

Linearity: $c_1x_1[n] + c_2x_2[n] \rightarrow \boxed{\mathcal{H}} \rightarrow c_1y_1[n] + c_2y_2[n]$

Together, additivity and homogeneity imply linearity.

Time-Invariance: $x[n - n_0] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n - n_0]$ for all n_0

Agenda for Recitation

- System properties: Linearity and time-invariance

What questions do you have from lecture?

Representations of LTI Systems

Linearity (L) and **time-invariance (TI)** are properties that greatly simplify the analysis and design of systems.

In this class, we will consider three equivalent representations of **discrete-time LTI systems**.

- linear, constant-coefficient difference equation
- unit-sample response $h[n]$
- frequency response $H(\Omega)$

At the same time, we will also consider three equivalent representations of **continuous-time LTI systems**.

- linear, constant-coefficient differential equation
- impulse response $h(t)$
- frequency response $H(\omega)$

Today, we will practice determining if a system defined by a difference (DT) or differential (CT) equation is **LTI**.

Discrete-Time Systems

Determine if each system is linear and time-invariant.
You may assume that each system is initially “at rest.”

$$x[n] \rightarrow \boxed{\mathcal{H}_1} \rightarrow y[n] = x[n] + 1$$

$$x[n] \rightarrow \boxed{\mathcal{H}_2} \rightarrow y[n] = x[2n]$$

$$x[n] \rightarrow \boxed{\mathcal{H}_3} \rightarrow y[n] = x[-n]$$

$$x[n] \rightarrow \boxed{\mathcal{H}_4} \rightarrow y[n] = y[n-1] + x[n]$$

$$x[n] \rightarrow \boxed{\mathcal{H}_5} \rightarrow y[n] = |x[n]|$$

$$x[n] \rightarrow \boxed{\mathcal{H}_6} \rightarrow y[n] = x[|n|]$$

$$x[n] \rightarrow \boxed{\mathcal{H}_7} \rightarrow y[n] = C \text{ (constant)}$$

Discrete-Time Systems

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$$x[n] \rightarrow \boxed{\mathcal{H}_1} \rightarrow y[n] = x[n] + 1$$

L? No. TI? Yes.

$$x[n] \rightarrow \boxed{\mathcal{H}_2} \rightarrow y[n] = x[2n]$$

L? Yes. TI? No.

$$x[n] \rightarrow \boxed{\mathcal{H}_3} \rightarrow y[n] = x[-n]$$

L? Yes. TI? No.

$$x[n] \rightarrow \boxed{\mathcal{H}_4} \rightarrow y[n] = y[n-1] + x[n]$$

L? Yes. TI? Yes.

$$x[n] \rightarrow \boxed{\mathcal{H}_5} \rightarrow y[n] = |x[n]|$$

L? No. TI? Yes.

$$x[n] \rightarrow \boxed{\mathcal{H}_6} \rightarrow y[n] = x[|n|]$$

L? Yes. TI? No.

$$x[n] \rightarrow \boxed{\mathcal{H}_7} \rightarrow y[n] = C \text{ (constant)}$$

L? Depends. TI? Yes.

Continuous-Time Systems

Determine if each system is linear and time-invariant.
You may assume that each system is initially “at rest.”

$$x(t) \rightarrow \boxed{\mathcal{H}_1} \rightarrow y(t) = \frac{d}{dt}x(t)$$

$$x(t) \rightarrow \boxed{\mathcal{H}_2} \rightarrow y(t) = \int_0^t x(\tau) d\tau$$

$$x(t) \rightarrow \boxed{\mathcal{H}_3} \rightarrow y(t) = \int_{-\infty}^t x(\tau) d\tau$$

$$x(t) \rightarrow \boxed{\mathcal{H}_4} \rightarrow y(t) = \int_t^{\infty} x(\tau) d\tau$$

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L? Yes. TI? Yes.

Common Confusion

A causal continuous-time (CT) system described by a linear, constant-coefficient differential equation (LCCDE) **with initial rest conditions** is linear and time-invariant.

$$\sum_k a_k \frac{d^k y(t)}{dt^k} = \sum_k b_k \frac{d^k x(t)}{dt^k}$$

A causal discrete-time (DT) system described by a linear, constant-coefficient difference equation (LCCDE) **with initial rest conditions** is linear and time-invariant.

$$\sum_k a_k y[n - k] = \sum_k b_k x[n - k]$$

A system specified by an LCCDE is **not necessarily LTI**. We need initial rest conditions to force the homogeneous solution to be identically zero. A system specified by an LCCDE **with initial rest conditions** is LTI.

Lessons Learned

Linearity (L) and **time-invariance (TI)** are properties that greatly simplify the analysis and design of systems.

Additivity: $x_1[n] + x_2[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y_1[n] + y_2[n]$

Homogeneity: $cx[n] \rightarrow \boxed{\mathcal{H}} \rightarrow cy[n]$ for any constant c

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Time-Invariance: $x[n - n_0] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n - n_0]$ for all n_0

Question of the Day

For the system below to be linear and time-invariant, what must be true about the constants M and B ?

$$x[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n] = Mx[n] + B$$