6.300: Signal Processing

Discrete-Time Fourier Transform

Analysis: $X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}$

Synthesis: $x[n]=\frac{1}{2\pi}\int_{2\pi}X(\Omega)e^{j\Omega n}d\Omega$

Time Delay: $x[n-n_0]\iff e^{-j\Omega n_0}X(\Omega)$

Frequency Derivative: $nx[n]\iff j\frac{d}{d\Omega}X(\Omega)$

Periodic Signals: $X(\Omega)=\sum_k 2\pi X[k]\delta(\Omega-k\Omega_0)$

Agenda for Recitation

- Discrete-time Fourier transform (DTFT)

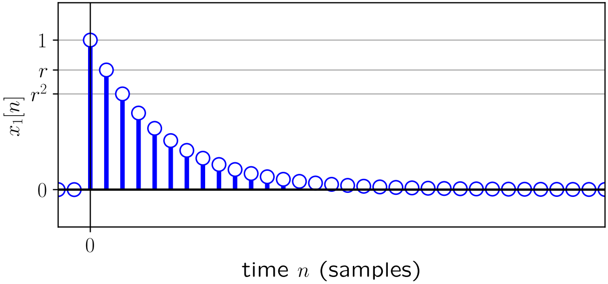
What questions do you have from lecture?

Fourier Transforms

Determine $X_1(\Omega)$, the Fourier transform of $x_1[n]$.

$$x_1[n] = \begin{cases} r^n & n \ge 0 \\ 0 & n < 0 \end{cases} \quad \text{where } |r| < 1$$

Sketch $|X_1(\Omega)|$ and $\angle X_1(\Omega)$.



Fourier Transforms

Determine $X_1(\Omega)$, the Fourier transform of $x_1[n]$.

$$x_1[n] = \begin{cases} r^n & n \geq 0 \\ 0 & n < 0 \end{cases} \text{ where } |r| < 1$$

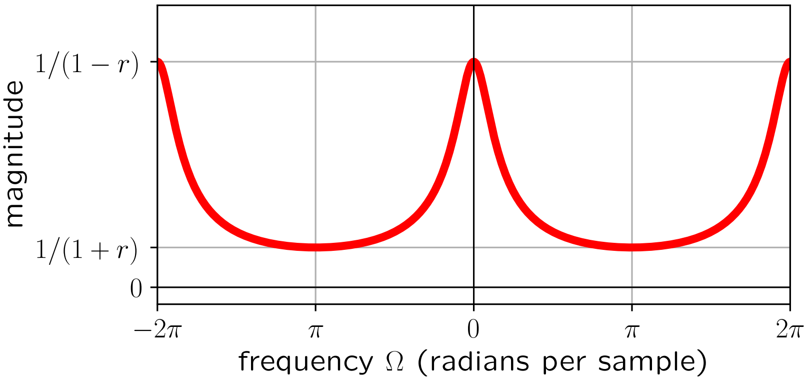
Sketch $|X_1(\Omega)|$ and $\angle X_1(\Omega)$.

Compute the Fourier transform.

$$X_1(\Omega) = \sum_{n=0}^{\infty} r^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(re^{-j\Omega}\right)^n = \frac{1}{1 - re^{-j\Omega}}$$

Sketch the magnitude and phase using a graphical method: Draw a complex-plane vector diagram.

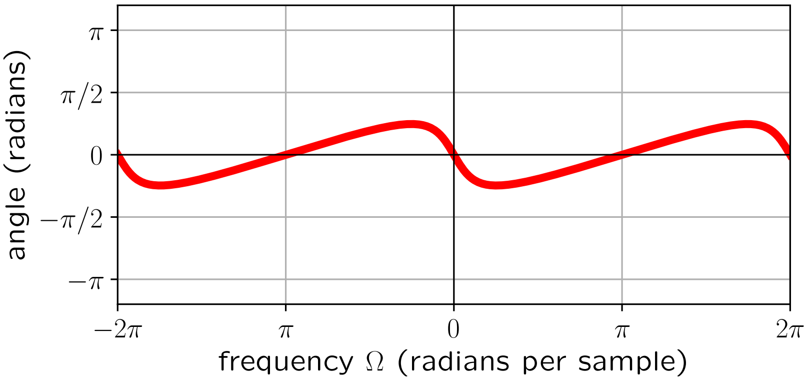
Fourier Transforms



The magnitude $|X_1(\Omega)|$ is shown above.

Because $e^{j\Omega}$ is periodic in 2π, so too is $|X_1(\Omega)|$.

Fourier Transforms



The angle $\angle X_1(\Omega)$ is shown above.

Because $e^{j\Omega}$ is periodic in 2π, so too is $\angle X_1(\Omega)$.

Fourier Transforms

Determine $X_1(\Omega)$, the Fourier transform of $x_1[n]$.

$$x_1[n] = \begin{cases} r^n & n \geq 0 \\ 0 & n < 0 \end{cases} \text{ where } |r| < 1$$

Sketch $|X_1(\Omega)|$ and $\angle X_1(\Omega)$.

As an aside, it's interesting to note that **geometric sequences** are the discrete-time (DT) analogue of continuous-time (CT) exponential functions.

$$e^{at}\Big|_{t=n\Delta} = e^{an\Delta} = \underbrace{\left(e^{a\Delta}\right)^n}_{r^n} = r^n$$

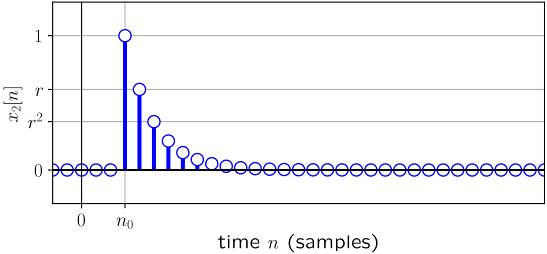
Exponential functions e^{at} solve differential equations, while geometric sequences r^n solve difference equations.

Fourier Transforms

Determine $X_2(\Omega)$, the Fourier transform of $x_2[n]$.

$$x_2[n] = x_1[n - n_0] = \begin{cases} r^{n-n_0} & n \geq n_0 \\ 0 & n < n_0 \end{cases}$$

Sketch $|X_2(\Omega)|$ and $\angle X_2(\Omega)$. How are $|X_2(\Omega)|$ and $|X_1(\Omega)|$ related? How are $\angle X_2(\Omega)$ and $\angle X_1(\Omega)$ related?



Fourier Transforms

Determine $X_2(\Omega)$, the Fourier transform of $x_2[n]$.

$$x_2[n] = x_1[n-n_0] = \begin{cases} r^{n-n_0} & n \ge n_0 \\ 0 & n < n_0 \end{cases}$$

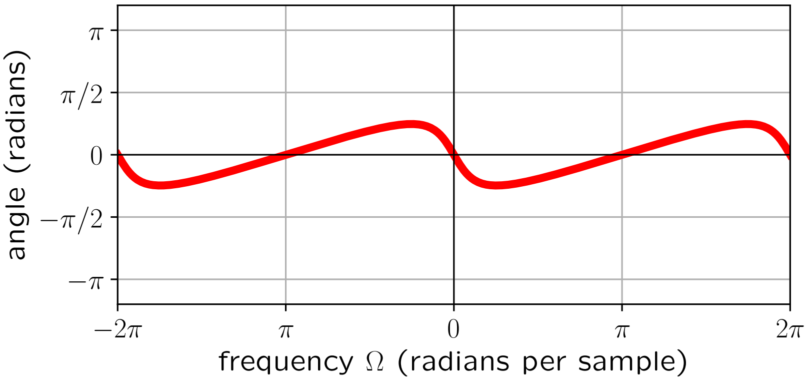
Sketch $|X_2(\Omega)|$ and $\angle X_2(\Omega)$. How are $|X_2(\Omega)|$ and $|X_1(\Omega)|$ related? How are $\angle X_2(\Omega)$ and $\angle X_1(\Omega)$ related?

Delay property: $x_2[n] = x_1[n-n_0] \iff X_2(\Omega) = e^{-j\Omega n_0} X_1(\Omega)$.

Magnitudes multiply: $|X_2(\Omega)| = |e^{-j\Omega n_0}||X_1(\Omega)| = |X_1(\Omega)|$.

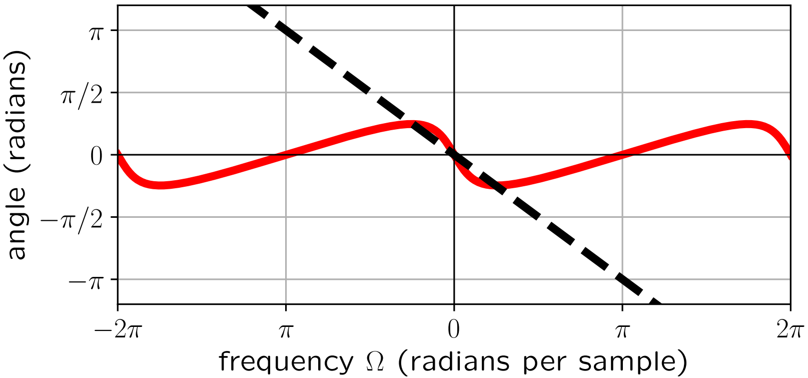
Angles add: $\angle X_2(\Omega) = \angle e^{-j\Omega n_0} + \angle X_1(\Omega) = -\Omega n_0 + \angle X_1(\Omega)$.

Fourier Transforms



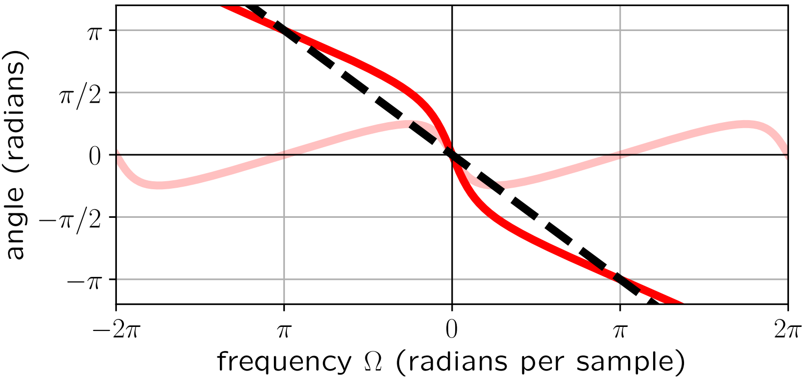
To sketch $\angle X_2(\Omega)$, start by sketching $\angle X_1(\Omega)$.

Fourier Transforms



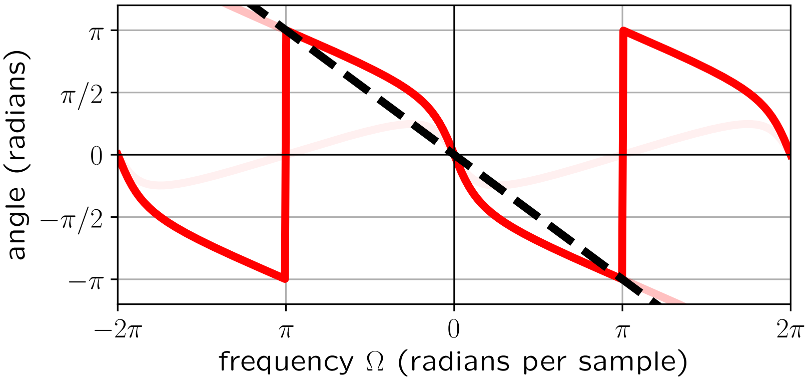
Sketch the $-\Omega n_0$ term.

Fourier Transforms



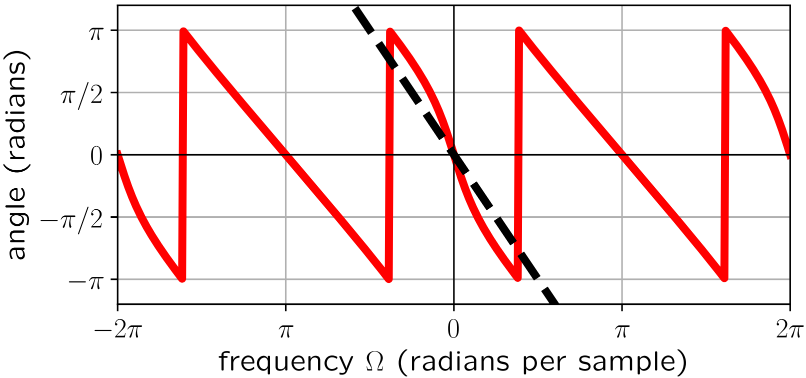
Angles add: $\angle X_2(\Omega) = -\Omega n_0 + \angle X_1(\Omega)$.

Fourier Transforms



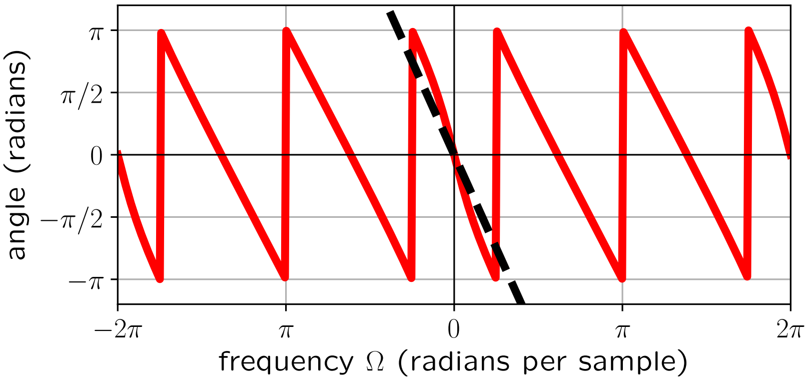
Finally, wrap $\angle X_2(\Omega)$ into the $[-\pi, \pi]$ range.

Fourier Transforms



As $n_0 \to \infty$, the $-\Omega n_0$ line slopes downward more steeply.

Fourier Transforms



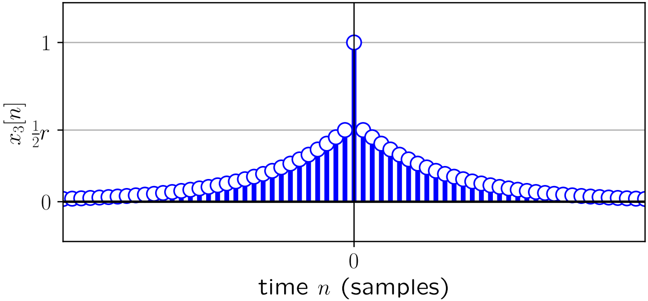
As $n_0 \to \infty$, the $-\Omega n_0$ line slopes downward more steeply.

Fourier Transforms

Determine $X_3(\Omega)$, the Fourier transform of $x_3[n]$.

$$x_3[n] = \text{Symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **symmetric**, what can you say about the Fourier transform?



Fourier Transforms

Determine $X_3(\Omega)$, the Fourier transform of $x_3[n]$.

$$x_3[n] = \text{Symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **symmetric**, what can you say about the Fourier transform?

The Fourier transform is linear.

$$x_3[n] = \tfrac{1}{2}x_1[n] + \tfrac{1}{2}x_1[-n]$$

$$X_3(\Omega) = \tfrac{1}{2}X_1(\Omega) + \tfrac{1}{2}X_1(-\Omega) = \frac{1 - r\cos(\Omega)}{1 - 2r\cos(\Omega) + r^2}$$

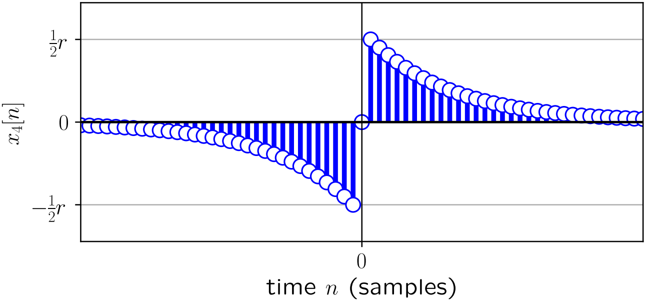
Notice that $X_3(\Omega)$ is real and symmetric.

Fourier Transforms

Determine $X_4(\Omega)$, the Fourier transform of $x_4[n]$.

$$x_4[n] = \text{Anti-symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **anti-symmetric**, what can you say about the Fourier transform?



Fourier Transforms

Determine $X_4(\Omega)$, the Fourier transform of $x_4[n]$.

$$x_4[n] = \text{Anti-symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **anti-symmetric**, what can you say about the Fourier transform?

The Fourier transform is linear.

$$x_4[n] = \tfrac{1}{2}x_1[n] - \tfrac{1}{2}x_1[-n]$$

$$X_4(\Omega) = \tfrac{1}{2}X_1(\Omega) - \tfrac{1}{2}X_1(-\Omega) = \frac{-jr\sin(\Omega)}{1-2r\cos(\Omega)+r^2}$$

Notice that $X_4(\Omega)$ is imaginary and anti-symmetric.

Fourier Transforms

Determine $X_5(\Omega)$, the Fourier transform of $x_5[n]$.

$$x_5[n] = n x_1[n]$$

Fourier Transforms

Determine $X_5(\Omega)$, the Fourier transform of $x_5[n]$.

$$x_5[n] = nx_1[n]$$

Differentiate in the frequency domain.

$$X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\Omega n}$$

$$\frac{d}{d\Omega}X_1(\Omega) = \sum_{n=-\infty}^{\infty} (-jn)x_1[n]e^{-j\Omega n}$$

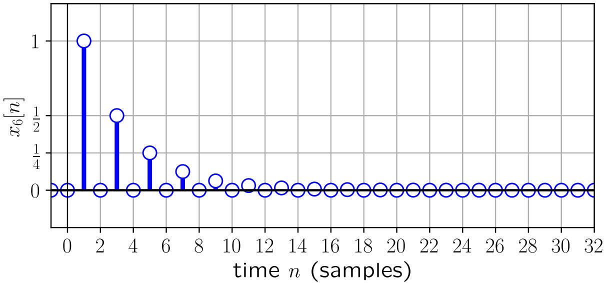
$$\underbrace{j\frac{d}{d\Omega}X_1(\Omega)}_{X_5(\Omega)} = \sum_{n=-\infty}^{\infty} \underbrace{(nx_1[n])}_{x_5[n]} e^{-j\Omega n} = \frac{re^{-j\Omega}}{(1-re^{-j\Omega})^2}$$

Fourier Transforms

Determine $X_6(\Omega)$, the Fourier transform of $x_6[n]$.

$$x_6[n] = \begin{cases} \left(\frac{1}{2}\right)^{\frac{n}{2}} & n \in \{0, 2, 4, 6, 8, 10, 12, \dots\} \\ 0 & \text{otherwise} \end{cases}$$

What does **"stretching in time"** do in frequency?



Fourier Transforms

Determine $X_6(\Omega)$, the Fourier transform of $x_6[n]$.

$$x_6[n] = \begin{cases} \left(\frac{1}{2}\right)^{\frac{n}{2}} & n \in \{0, 2, 4, 6, 8, 10, 12, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

What does **"stretching in time"** do in frequency?

Make a change of variables: $n = 2m$.

$$X_6(\Omega) = \sum_{n \text{ even}}^{\infty} \left(\tfrac{1}{2}\right)^{\frac{n}{2}} e^{-j\Omega n} = \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^{m} e^{-j\Omega(2m)} = \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\, e^{-j2\Omega}\right)^{m}$$

Stretching in time? Compressing in frequency!

$$X_6(\Omega) = \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\, e^{-j2\Omega}\right)^{m} = \frac{1}{1 - \frac{1}{2}\, e^{-j2\Omega}}$$

Fourier Transforms

$x_7[n]$ has Fourier transform $X_7(\Omega)$. Determine $x_7[n]$.

$$X_7(\Omega) = e^{-j3\Omega}$$

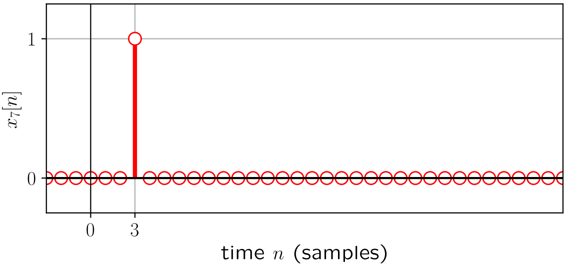
Fourier Transforms

$x_7[n]$ has Fourier transform $X_7(\Omega)$. Determine $x_7[n]$.

$$X_7(\Omega) = e^{-j3\Omega}$$

Plug in to the synthesis equation.

$$x_7[n] = \frac{1}{2\pi}\int_{2\pi} e^{-j3\Omega} e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} e^{j\Omega(n-3)} d\Omega = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases}$$



Lessons Learned

The **discrete-time Fourier transform (DTFT)** is a Fourier representation for aperiodic and periodic discrete-time signals. It has many useful properties.

Analysis: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

Synthesis: $x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$

Time Delay: $x[n-n_0] \iff X(\Omega) e^{-j\Omega n_0}$

Frequency Derivative: $nx[n] \iff j\frac{d}{d\Omega} X(\Omega)$

Periodic Signals: $X(\Omega) = \sum_k 2\pi X[k] \delta(\Omega - k\Omega_0)$

Question of the Day

Previously, we derived the **time-derivative property** for the continuous-time Fourier transform.

$$\frac{d}{dt}x(t) \Longleftrightarrow j\omega X(\omega)$$

Does the **discrete-time Fourier transform** have a time-derivative property? Why or why not?