

## 6.300: Signal Processing

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### Fourier Series II: Symmetry

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right)$$

The **analysis equations** tell us how to calculate these Fourier series coefficients.

$$c_0 = \frac{1}{T} \int_T f(t) dt \quad \text{average value over a period}$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

$$d_k = \frac{2}{T} \int_T f(t) \sin\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

# Agenda for Recitation

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- Definitions of **symmetry**, **anti-symmetry**, **asymmetry**
- Using **symmetry** to simplify Fourier series calculations

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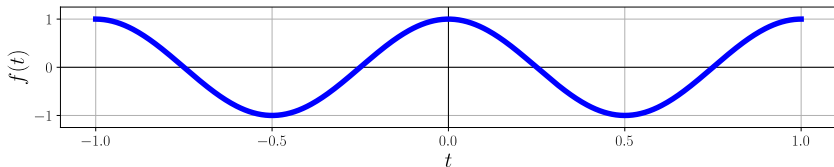
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What questions do you have from lecture?

# Symmetry, Anti-Symmetry, Asymmetry

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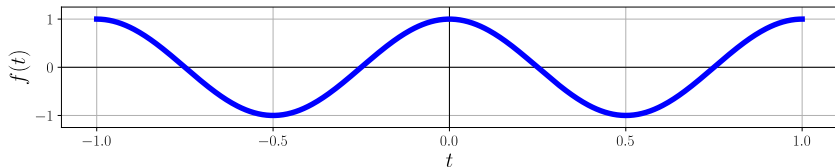
A **symmetric** signal satisfies  $f(t) = f(-t)$ .



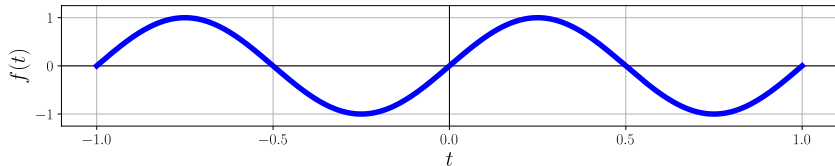
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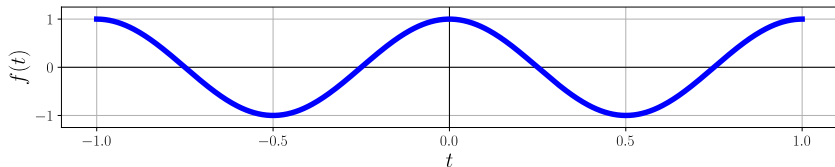
An **anti-symmetric** signal satisfies  $f(t) = -f(-t)$ .



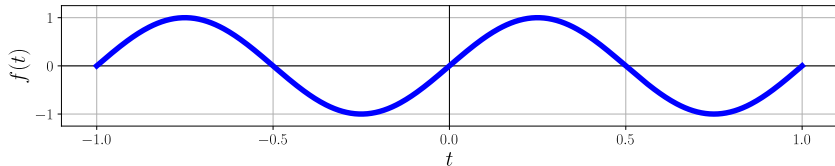
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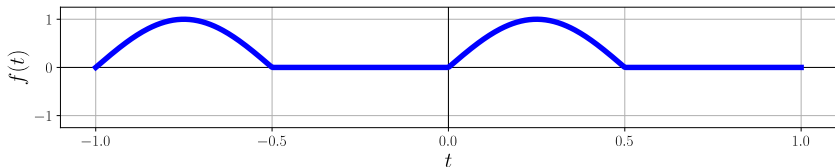


What we've called **symmetry/anti-symmetry** is also called **even/odd symmetry**. Might as well know both.

# Symmetry, Anti-Symmetry, Asymmetry

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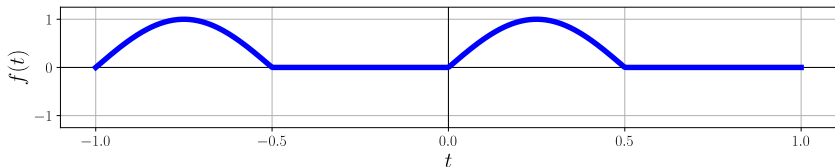
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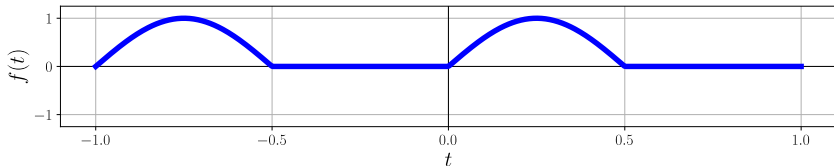
We can express an **asymmetric** signal as the **sum** of a **symmetric part** and an **anti-symmetric part**.

$$f(t) = f_{\text{symmetric}}(t) + f_{\text{anti-symmetric}}(t)$$

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$$f(t) = f_{\text{symmetric}}(t) + f_{\text{anti-symmetric}}(t)$$

How do we get  $f_{\text{symmetric}}(t)$  from  $f(t)$ ?

How do we get  $f_{\text{anti-symmetric}}(t)$  from  $f(t)$ ?

# Symmetry, Anti-Symmetry, Asymmetry

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How do we get  $f_{\text{symmetric}}(t)$  from  $f(t)$ ?

How do we get  $f_{\text{anti-symmetric}}(t)$  from  $f(t)$ ?

Add  $f(t)$  and  $f(-t)$  to cancel out the anti-symmetric part.  
This leaves the symmetric part.

$$f_{\text{symmetric}}(t) = \frac{f(t) + f(-t)}{2}$$

Add  $f(t)$  and  $-f(-t)$  to cancel out the symmetric part.  
This leaves the anti-symmetric part.

$$f_{\text{anti-symmetric}}(t) = \frac{f(t) - f(-t)}{2}$$

# Symmetry, Anti-Symmetry, Asymmetry

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Before we make any calculations, we can use **symmetry** (and **anti-symmetry**) to make some conclusions about the Fourier series coefficients.

# Symmetry, Anti-Symmetry, Asymmetry

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Before we make any calculations, we can use **symmetry** (and **anti-symmetry**) to make some conclusions about the Fourier series coefficients.

What can we say about the Fourier series coefficients ( $c_k$  and  $d_k$ ) for a **real, symmetric** signal?

What can we say about the Fourier series coefficients ( $c_k$  and  $d_k$ ) for a **real, anti-symmetric** signal?

# Symmetry, Anti-Symmetry, Asymmetry

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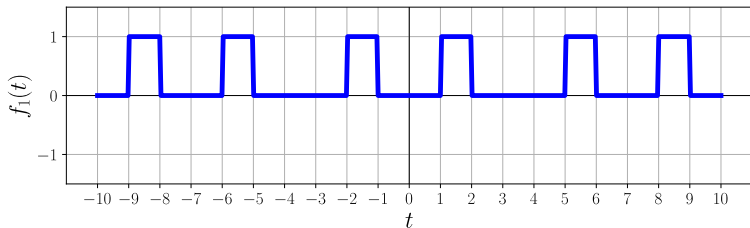
What can we say about the Fourier series coefficients ( $c_k$  and  $d_k$ ) for a **real, symmetric** signal?

What can we say about the Fourier series coefficients ( $c_k$  and  $d_k$ ) for a **real, anti-symmetric** signal?

Instead of keeping the discussion abstract, let's work through examples and draw some conclusions from there.

# Fourier Series and Symmetry

Determine the Fourier series coefficients ( $c_k$  and  $d_k$ ) for the periodic function  $f_1(t)$ , shown below.



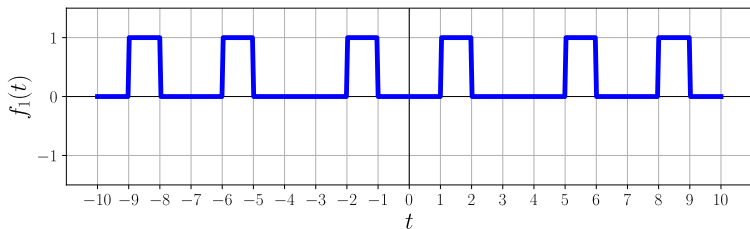
What is the fundamental period ( $T$ ) of  $f_1(t)$ ?

Does it matter which interval you integrate over?  
(0 to  $T$ ?  $-\frac{1}{2}T$  to  $\frac{1}{2}T$ ?  $\frac{5}{16}T$  to  $\frac{21}{16}T$ ?)

Do you notice anything about the  $c_k$  and/or  $d_k$  terms?

# Fourier Series and Symmetry

Determine the Fourier series coefficients ( $c_k$  and  $d_k$ ) for the periodic function  $f_1(t)$ , shown below.



The fundamental period is  $T = 7$ .

Integrating over any length- $T$  interval yields the same result. (The calculations are on the next slide.)

Because  $f_1(t)$  is symmetric,  $d_k = 0$  for all  $k$ .

# Fourier Series and Symmetry

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$c_0$  is the average value of  $f_1(t)$  over one period.

$$c_0 = \frac{1}{7} \int_{-\frac{7}{2}}^{\frac{7}{2}} f_1(t) dt = \frac{2}{7}$$

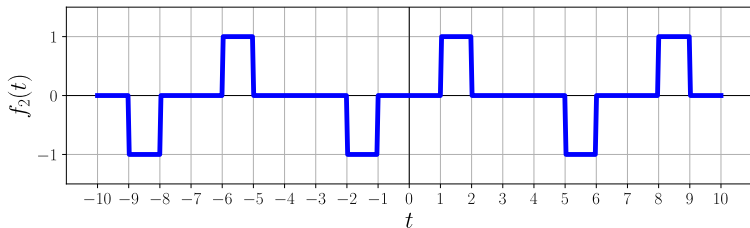
Compute the remaining coefficients via integration.

$$c_k = \frac{1}{7} \int_{-\frac{7}{2}}^{\frac{7}{2}} f_1(t) \cos\left(k \frac{2\pi}{7} t\right) dt = \frac{2}{\pi k} \left( \sin\left(\frac{4\pi}{7} k\right) - \sin\left(\frac{2\pi}{7} k\right) \right)$$

$$d_k = \frac{1}{7} \int_{-\frac{7}{2}}^{\frac{7}{2}} f_1(t) \sin\left(k \frac{2\pi}{7} t\right) dt = 0$$

# Fourier Series and Symmetry

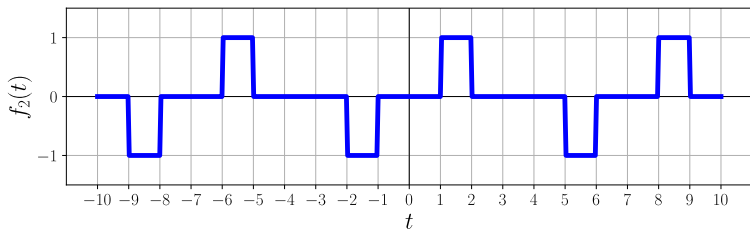
Determine the Fourier series coefficients ( $c_k$  and  $d_k$ ) for the periodic function  $f_2(t)$ , shown below.



Do you notice anything about the  $c_k$  and/or  $d_k$  terms?

# Fourier Series and Symmetry

Determine the Fourier series coefficients ( $c_k$  and  $d_k$ ) for the periodic function  $f_2(t)$ , shown below.



Do you notice anything about the  $c_k$  and/or  $d_k$  terms?

Because  $f_2(t)$  is anti-symmetric,  $c_k = 0$  for all  $k$ .

# Fourier Series and Symmetry

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$c_0$  is the average value of  $f_2(t)$  over one period.

$$c_0 = \frac{1}{7} \int_{-\frac{7}{2}}^{\frac{7}{2}} f_2(t) dt = 0$$

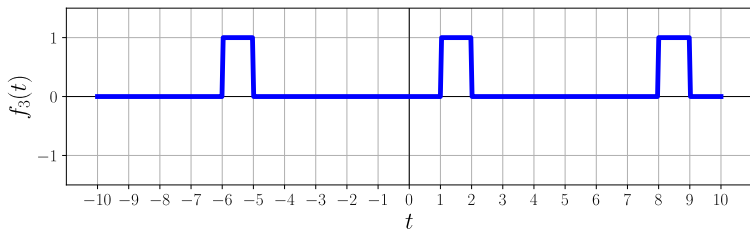
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$$c_k = \frac{1}{7} \int_{-\frac{7}{2}}^{\frac{7}{2}} f_2(t) \cos\left(k \frac{2\pi}{7} t\right) dt = 0$$

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# Fourier Series and Symmetry

Determine the Fourier series coefficients ( $c_k$  and  $d_k$ ) for the periodic function  $f_3(t)$ , shown below.



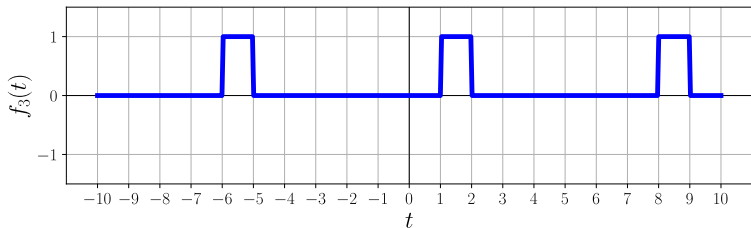
Do you notice anything about the  $c_k$  and/or  $d_k$  terms?

How does  $f_3(t)$  relate to  $f_1(t)$  and  $f_2(t)$ ?

How do the Fourier series coefficients for  $f_3(t)$  relate to the coefficients for  $f_1(t)$  and  $f_2(t)$ ?

# Fourier Series and Symmetry

Determine the Fourier series coefficients ( $c_k$  and  $d_k$ ) for the periodic function  $f_3(t)$ , shown below.



$f_3(t)$  is asymmetric. Both  $c_k$  and  $d_k$  are non-zero.

Notice that  $f_3(t) = \frac{1}{2}f_1(t) + \frac{1}{2}f_2(t)$ . So, the coefficients for  $f_3(t)$  are half those for  $f_1(t)$  plus half those for  $f_2(t)$ .

# Fourier Series and Symmetry

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$c_0$  is the average value of  $f_3(t)$  over one period.

$$c_0 = \frac{1}{7} \int_{-\frac{7}{2}}^{\frac{7}{2}} f_3(t) dt = \frac{1}{7}$$

Compute the remaining coefficients via integration.

$$c_k = \frac{1}{7} \int_{-\frac{7}{2}}^{\frac{7}{2}} f_3(t) \cos\left(k \frac{2\pi}{7} t\right) dt = \frac{1}{\pi k} \left( \sin\left(\frac{4\pi}{7} k\right) - \sin\left(\frac{2\pi}{7} k\right) \right)$$

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# Fourier Series and Symmetry

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We have looked at how to use **symmetry** to simplify calculations with Fourier series.

**Real, symmetric** signals: Only  $c_k$  are non-zero.

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We will soon learn many more **Fourier properties**.

e.g., If we know  $c_k/d_k$  for  $f(t)$ , what about  $c_k/d_k$  for ...

- $f(t-1)$ ?
- $f(2t)$ ?
- $f(t)\cos(t)$ ?
- $f^2(t)$ ?

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**Next time:** Simplifying the math even further using **complex numbers** — **replacing trig with algebra!**

# Lessons Learned

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The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right)$$

The **analysis equations** tell us how to calculate these Fourier series coefficients.

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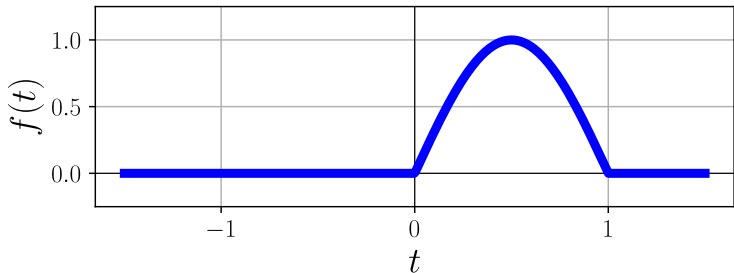
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**Real, anti-symmetric** signals: Only  $d_k$  are non-zero.

## Question of the Day

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An **asymmetric** function  $f(t)$  is shown below.  
The function is zero outside the indicated bounds.



Sketch  $f_{\text{symmetric}}(t)$ , the **symmetric part**.

Sketch  $f_{\text{anti-symmetric}}(t)$ , the **anti-symmetric part**.

