

6.300: Signal Processing

Fourier Series I: Analysis and Synthesis

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right)$$

The **analysis equations** tell us how to calculate these Fourier series coefficients.

$$c_0 = \frac{1}{T} \int_T f(t) dt \quad \text{average value over a period}$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

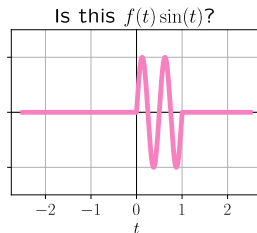
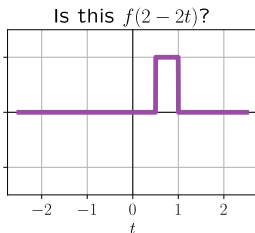
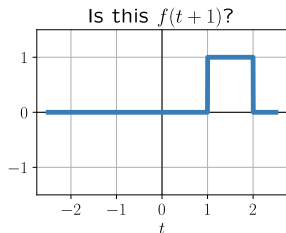
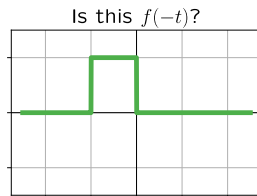
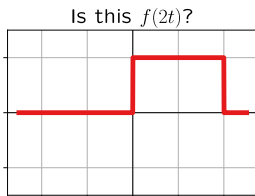
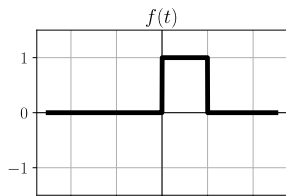
$$d_k = \frac{2}{T} \int_T f(t) \sin\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

Agenda for Recitation

- Signals (and **transformations** thereof)
- **Fourier series** expansion for periodic signals

Transformations

How many expressions match the plot beneath them?

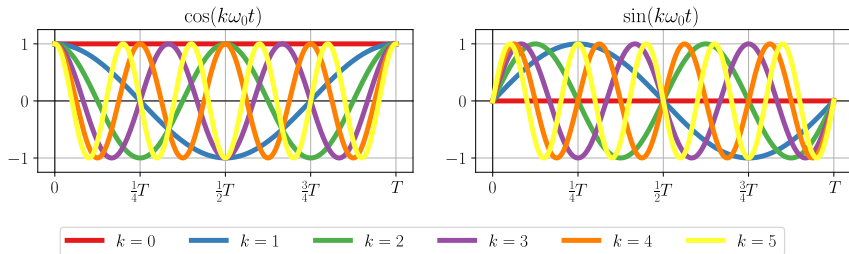


Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t + T)$ using a **Fourier series** expansion.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right)$$

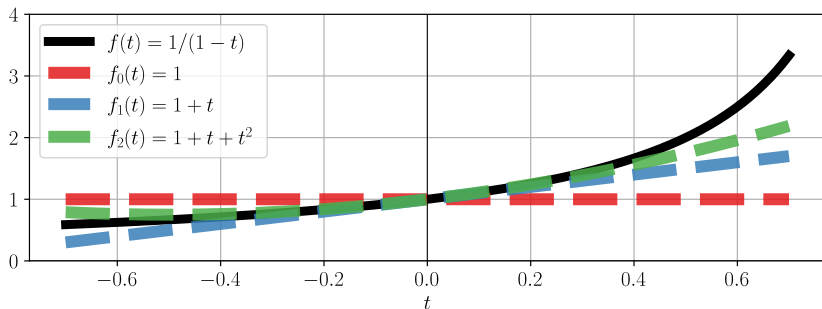
A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.



Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

$$f(t) = \frac{1}{1-t} \approx 1 + t + t^2 \text{ for } t \approx 0$$



Series Expansions

Polynomials are simple to work with — hence the utility of Taylor series. So, what's the use of Fourier methods?

Why Fourier? Why not Taylor?

Fourier Series

A **Fourier series** expansion takes the form

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right).$$

How do we determine the **coefficients**, though?

In lecture, Professor Freeman derived formulæ by exploiting the **orthogonality** (perpendicularity) of harmonically-related sinusoids over a length- T interval.

$$c_0 = \frac{1}{T} \int_T f(t) dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(k \frac{2\pi}{T} t\right) dt \quad (\text{for } k \geq 1)$$

$$d_k = \frac{2}{T} \int_T f(t) \sin\left(k \frac{2\pi}{T} t\right) dt \quad (\text{for } k \geq 1)$$

Fourier Series

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Let $f(t) = f(t+2\pi) = \cos(t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right).$$

Sums of Sinusoids

A **Fourier series** expansion takes the form

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right).$$

How many of the functions below have **only one** non-zero Fourier series coefficient?

- $f_1(t) = \cos\left(t - \frac{\pi}{2}\right)$
- $f_2(t) = \cos^2(t)$
- $f_3(t) = \sin(t) \cos(t)$
- $f_4(t) = 4 \cos^3(t) - 3 \cos(t)$
- $f_5(t) = \cos(12t) \cos(4t) \cos(2t)$

Hint: Do you need to integrate?

Trigonometric identities are given on the next slide.

Sums of Sinusoids

Trigonometric Identities

(not comprehensive)

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2} \cos(2\theta)$$

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2} \cos(2\theta)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$$

$$\cos(\alpha) \cos(\beta) = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)$$

$$\sin(\alpha) \sin(\beta) = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta)$$

$$\sin(\alpha) \cos(\beta) = \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta)$$

$$\cos(\alpha) \sin(\beta) = \frac{1}{2} \sin(\alpha + \beta) - \frac{1}{2} \sin(\alpha - \beta)$$

Don't worry if your trigonometry is rusty. We'll soon find a way to replace all this trigonometry with simple algebra.

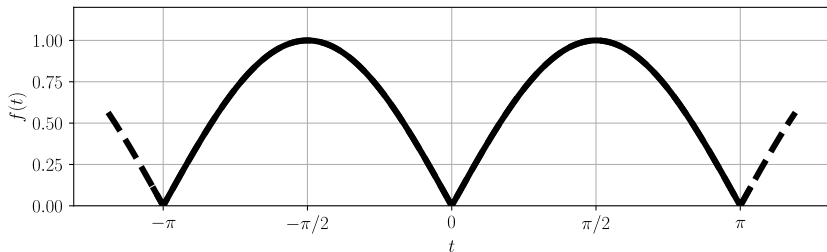
Periodicity

Let $f(t) = \cos(t) + \cos(2\pi t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right).$$

If such an expansion is not possible, explain why.

Rectified Sine



Let $f(t) = |\sin(t)|$, as shown above.

First, estimate the value of c_0 graphically.

Then determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right).$$

Lessons Learned

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

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Do not mindlessly plug numbers into formulas!

Think before you calculate!