

6.300: Signal Processing

Fourier Series I: Analysis and Synthesis

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right)$$

The **analysis equations** tell us how to calculate these Fourier series coefficients.

$$c_0 = \frac{1}{T} \int_T f(t) dt \quad \text{average value over a period}$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

$$d_k = \frac{2}{T} \int_T f(t) \sin\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

Agenda for Recitation

- Signals (and **transformations** thereof)
- **Fourier series** expansion for periodic signals

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What are you hoping to get out of 6.300?

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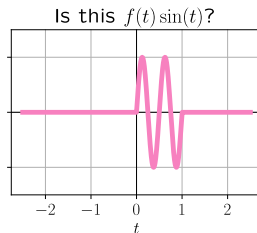
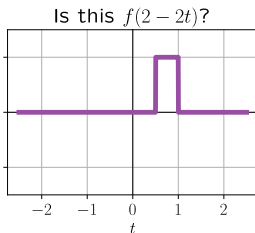
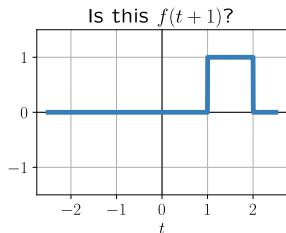
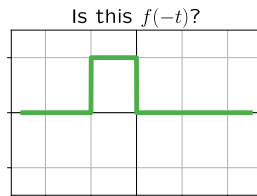
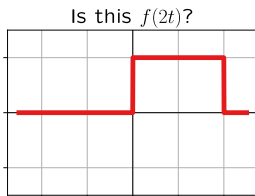
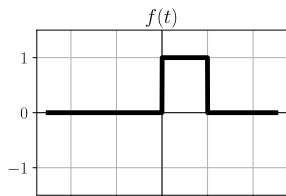
What are you hoping to get out of 6.300?

Fourier methods are fundamental to signal processing — and to many other fields of science and engineering!

- acoustics, electromagnetics, and optics
- audio, speech, and music processing
- biomedicine (ECG, EEG, MRI)
- communications
- control
- oceanography, radio astronomy, and seismology
- remote sensing (RADAR, SONAR, LIDAR)

Transformations

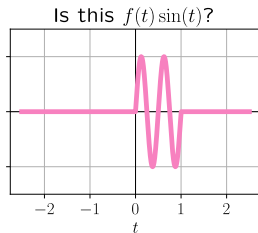
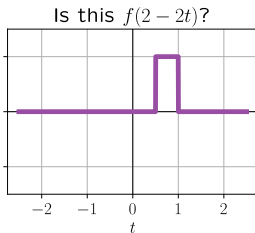
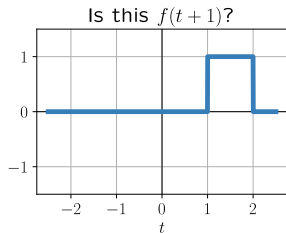
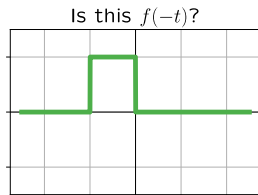
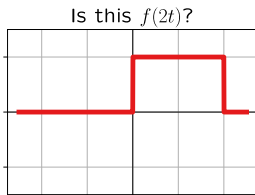
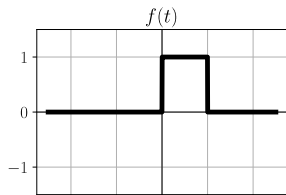
How many expressions match the plot beneath them?



Transformations

How many expressions match the plot beneath them?

Answer: 2 $\rightarrow f(-t)$ and $f(2 - 2t)$



Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t + T)$ using a **Fourier series** expansion.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right)$$

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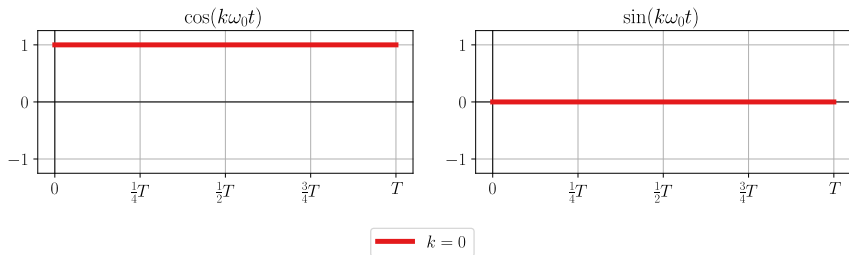
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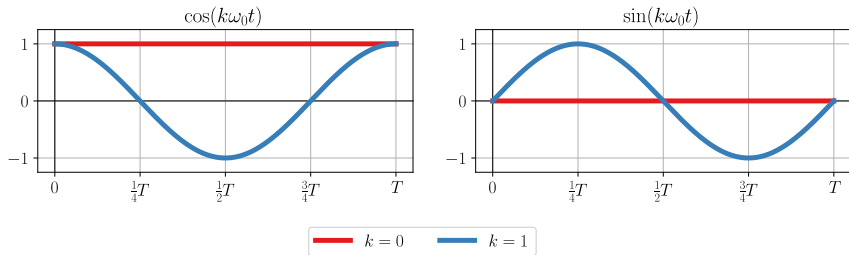


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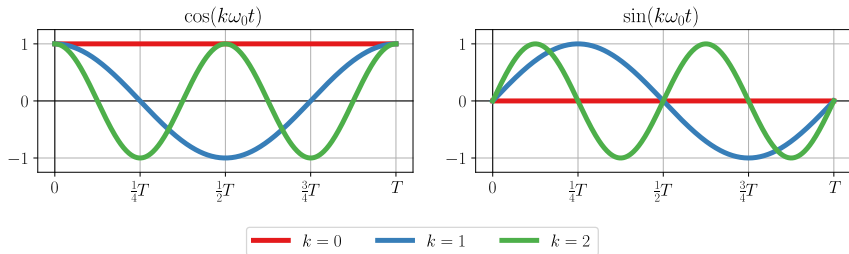


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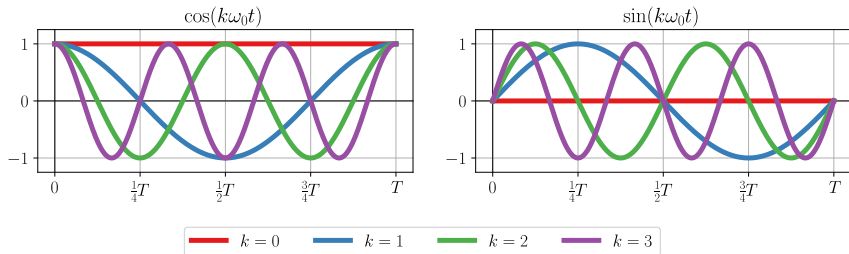


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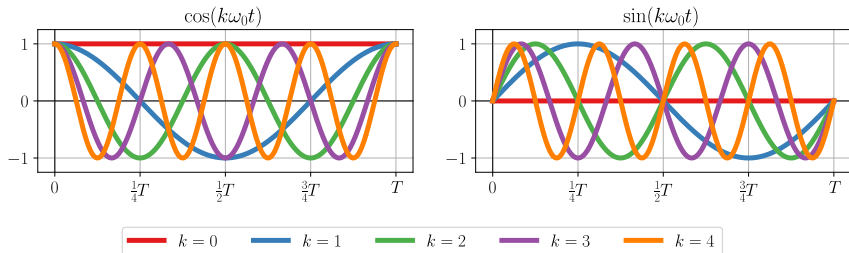


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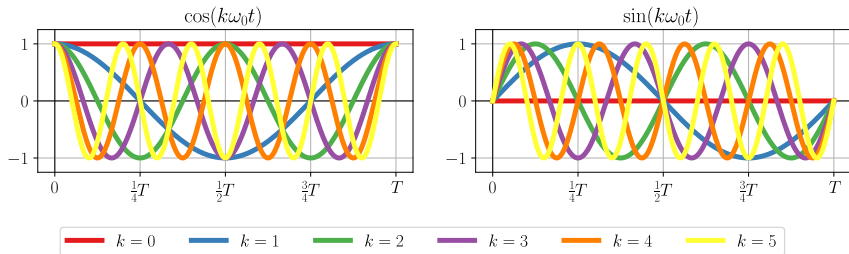


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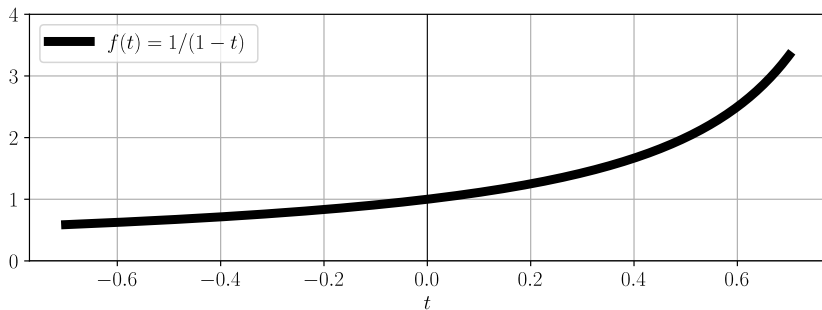
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Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

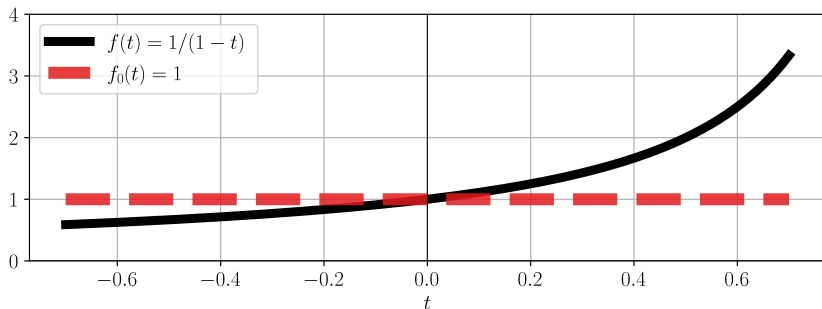
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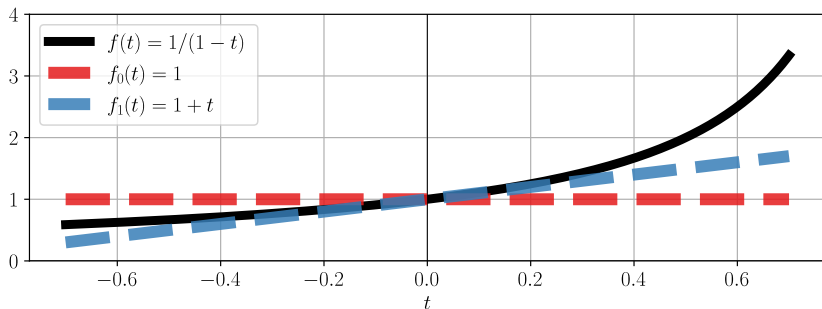
$$f(t) = \frac{1}{1-t} \approx 1 \text{ for } t \approx 0$$



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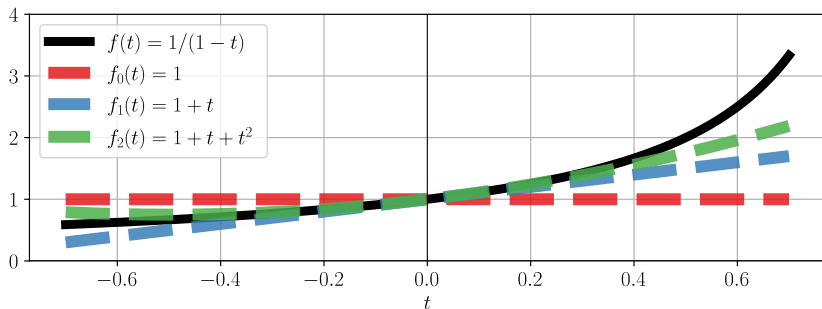
$$f(t) = \frac{1}{1-t} \approx 1 + t \text{ for } t \approx 0$$



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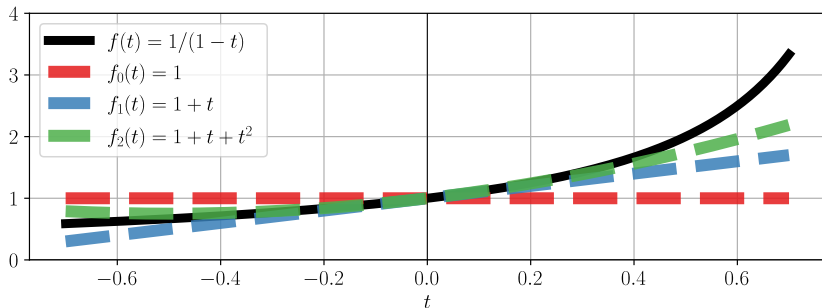
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(No, you don't need to know this for the quiz.)

Series Expansions

Polynomials are simple to work with — hence the utility of Taylor series. So, what's the use of Fourier methods?

Why Fourier? Why not Taylor?

Fourier Series

A **Fourier series** expansion takes the form

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right).$$

How do we determine the **coefficients**, though?

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How do we determine the **coefficients**, though?

In lecture, Professor Freeman derived formulæ by exploiting the **orthogonality** (perpendicularity) of harmonically-related sinusoids over a length- T interval.

$$c_0 = \frac{1}{T} \int_T f(t) dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(k \frac{2\pi}{T} t\right) dt \quad (\text{for } k \geq 1)$$

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Let $f(t) = f(t+2\pi) = \cos(t)$. Determine the coefficients for a Fourier series expansion of the form

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$c_1 = 1$. All other coefficients are zero.

Sums of Sinusoids

A **Fourier series** expansion takes the form

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How many of the functions below have **only one** non-zero Fourier series coefficient?

- $f_1(t) = \cos\left(t - \frac{\pi}{2}\right)$
- $f_2(t) = \cos^2(t)$
- $f_3(t) = \sin(t) \cos(t)$
- $f_4(t) = 4 \cos^3(t) - 3 \cos(t)$
- $f_5(t) = \cos(12t) \cos(4t) \cos(2t)$

Hint: Do you need to integrate?

Trigonometric identities are given on the next slide.

Sums of Sinusoids

Trigonometric Identities

(not comprehensive)

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2} \cos(2\theta)$$

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2} \cos(2\theta)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$$

$$\cos(\alpha) \cos(\beta) = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)$$

$$\sin(\alpha) \sin(\beta) = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta)$$

$$\sin(\alpha) \cos(\beta) = \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta)$$

$$\cos(\alpha) \sin(\beta) = \frac{1}{2} \sin(\alpha + \beta) - \frac{1}{2} \sin(\alpha - \beta)$$

Don't worry if your trigonometry is rusty. We'll soon find a way to replace all this trigonometry with simple algebra.

Sums of Sinusoids

How many of the functions below have **only one** non-zero Fourier series coefficient? **3** $\rightarrow f_1, f_3, f_4$

- $f_1(t) = \cos\left(t - \frac{\pi}{2}\right) = \sin(t)$
- $f_2(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2} \cos(2t)$
- $f_3(t) = \sin(t) \cos(t) = \frac{1}{2} \sin(2t)$
- $f_4(t) = 4 \cos^3(t) - 3 \cos(t) = \cos(3t)$
- $f_5(t) = \dots = \frac{1}{4} \cos(6t) + \frac{1}{4} \cos(10t) + \frac{1}{4} \cos(14t) + \frac{1}{4} \cos(18t)$

Periodicity

Let $f(t) = \cos(t) + \cos(2\pi t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \frac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \frac{2\pi}{T} t\right).$$

If such an expansion is not possible, explain why.

Periodicity

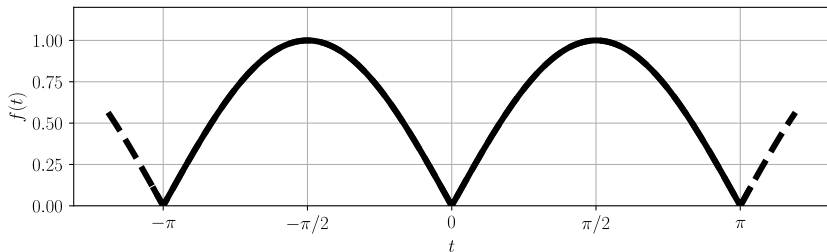
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If such an expansion is not possible, explain why.

$\cos(t)$ is periodic in $T_1 = 2\pi$, while $\cos(2\pi t)$ is periodic in $T_2 = 1$. There is no $T > 0$ for which $\cos(t) = \cos(t + T)$ and $\cos(2\pi t) = \cos(2\pi(t + T))$ for all t . That is, $f(t)$ is **aperiodic** — and hence $f(t)$ has no Fourier series representation.

Rectified Sine



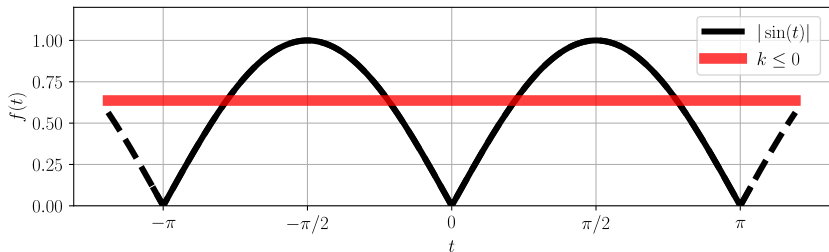
Let $f(t) = |\sin(t)|$, as shown above.

First, estimate the value of c_0 graphically.

Then determine the coefficients for a Fourier series expansion of the form

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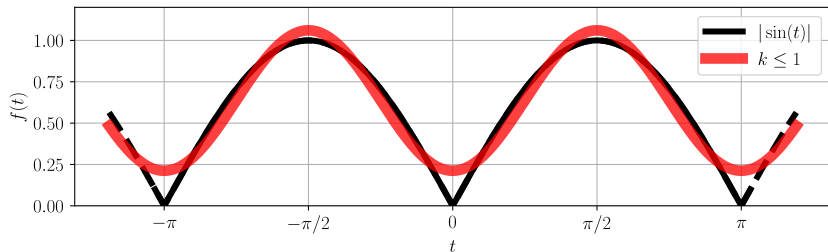
c_0 is the average value over one period. It looks like the value of c_0 should be somewhere between 0.5 and 1.

$$c_0 = \frac{1}{T} \int_T f(t) dt = \frac{1}{\pi} \int_0^\pi \sin(t) dt = \frac{2}{\pi} \approx 0.64$$

The function is symmetric about $t = 0$, so $d_k = 0$ for all k .

$$c_k = \frac{2}{\pi} \int_0^\pi \sin(t) \cos(2kt) dt = \frac{-4/\pi}{4k^2 - 1} \text{ for } k \geq 1$$

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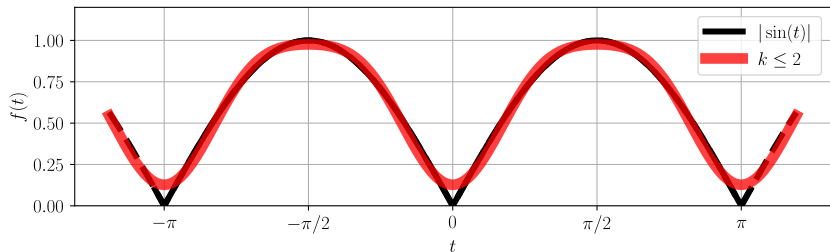
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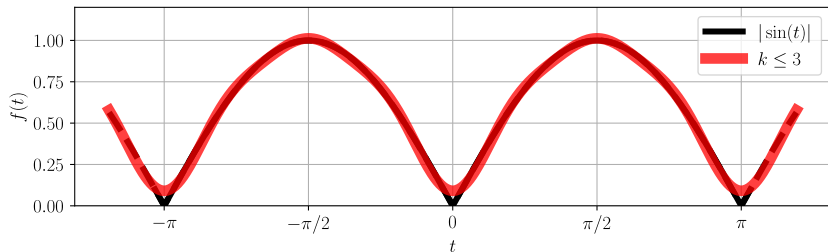
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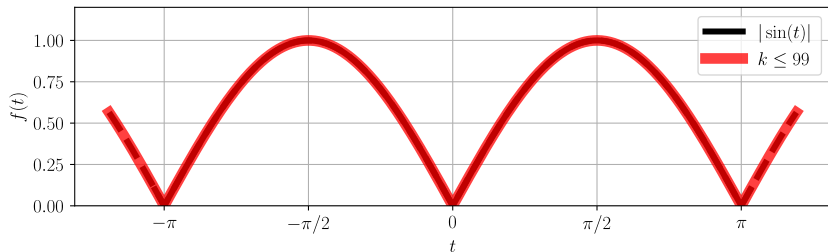
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Lessons Learned

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The **analysis equations** tell us how to calculate these Fourier series coefficients.

$$c_0 = \frac{1}{T} \int_T f(t) dt \quad \text{average value over a period}$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

$$d_k = \frac{2}{T} \int_T f(t) \sin\left(k \frac{2\pi}{T} t\right) dt \quad \text{for } k \geq 1$$

Do not mindlessly plug numbers into formulas!

Think before you calculate!

Question of the Day

Let $f(t) = \cos^2(t - \frac{\pi}{2})$.

Make a rough sketch of $f(t)$.

What is the fundamental period (T)?

What is the fundamental frequency (ω_0)?

Without resorting to calculus, determine coefficients for a Fourier series expansion of the form

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t).$$