ES.S20: Group Theory

Worksheet #4

Problem #1

(i) Write down an example of a group of order m, where $m \in \{9, 10, 11, 12, 13, 14, 15\}$.

(ii) Write down all that you know about groups of order $m \in \{9, 10, 11, 12, 13, 14, 15\}$.

(i) We can always write down the cyclic group of any order m abstractly as

$$\{a^r \mid a^m = 1\} \text{ or } \langle a \mid a^m = 1 \rangle$$

which is realized concretely as

$$\langle \theta \mid \theta \text{ denotes a rotation of an } m\text{-spoked wheel through } 2\pi/m \text{ radians} \rangle.$$

(ii) This is open-ended, but let us emphasize the following.

Groups of order p for p prime: In class, we showed that if $|G| = p$ for some prime p, then the group G is always cyclic of order p. There are no other groups of this order. So, 11 and 13 are prime and such a group of order 11 or order 13 is cyclic.

Groups of order p^2 for p prime: $9 = p^2$ for a prime $p = 3$. If $a \in G$ and $|G| = 9$, then the order of a is either 3 or 9. If the only elements are of order 3 and no elements are of order 9, then the group G is the direct product of two subgroups G_1 and G_2.

$$G_1 = \langle a \mid a^3 = 1 \rangle \qquad G_2 = \langle b \mid b^3 = 1 \rangle \qquad G = G_1 \times G_2$$

If there an element of order 9, then G is cyclic and generated by an element of order 9.

Groups of order pq for p, q prime: Let's look at $m = 15$ for a concrete example. There is a result that says that if $|G| = pq$ and p, q (where $p > q$) are prime, then the subgroup of order p is normal.

$$H = \langle b \mid b^p = 1 \rangle \qquad K = \langle a \mid a^q = 1 \rangle$$

So, we have a subgroup $H \triangleleft G$ so $a^{-1}ba \in H$ so $a^{-1}ba = b^r$. What are the constraints on r? Assume $ab \neq ba$, so $r \not\equiv 1 \bmod p$.

$$a^{-2}ba^2 = a^{-1}(a^{-1}ba)a = a^{-1}(b^r a) = (a^{-1}ba)^r = b^{r^2}$$

$$a^{-q}ba^q = b^{r^q} \text{ but } a^q = 1 \text{ so } b = b^{r^q}$$

$$r^q = 1 \bmod p \implies q \,|\, p-1$$

So, there is another non-trivial subgroup only if q divides $p-1$.

For example, if $m = 10$, then $10 = 5 \times 2$ which implies that 2 divides $5 - 1 = 4$. So, for a group of order 10, we have the following subgroups of order $p = 5$ and $q = 2$, respectively.

$$H = \langle b \,|\, b^5 = 1 \rangle \qquad K = \langle a \,|\, a^2 = 1 \rangle$$

If a, b commute then $(ab)^{10} = 1$ and G must be cyclic. If $a^{-1}ba = b^r$ then $r^q = 1 \bmod p$ and we claim that we have a solution other than $r = 1$. We can test out $r \in \{1, 2, 3, 4\}$. $r = 1$ works, $r = 2$ does not, $\ldots r = 4$ yields a solution, as $4^2 = 16 = 1 \bmod 5$. So, we have a group of order 10 given by $\langle a, b \,|\, a^2 = 1, b^5 = 1, a^{-1}ba = b^4 \rangle$.

Let's examine a group of order $m = 15 = 5 \times 3$ where $p = 5$ and $q = 3$. 3 does not divide $5 - 1 = 4$, so the only solution is as follows.

$$H = \langle b \,|\, b^5 = 1 \rangle \qquad K = \langle a \,|\, a^3 = 1 \rangle$$

If $a^{-1}ba = b^r$ then the only solution is $r = 1$. So, $ab = ba$, and ab is an element of order 15, so G is always the cyclic group.

Groups of order p^2q for p, q prime: The last group we'll examine is a group of order $m = 12 = 2^2 \times 3 = p^2q$ for $p = 2$ and $q = 3$. There is always the cyclic group of order 12, of course. We also know that the alternating group A_4 has order $|A_4| = 12$. We will not analyze this further in this seminar. The group is of order p^2q, so we cannot apply the arguments regarding a group of order pq from above.

Problem #2

(i) Using the cube given to you, count the number of faces, vertices, and edges.

(ii) Convince yourself that a rotation of the cube must be about an axis through the center of the cube. These axes are listed below.

- axis through the center of opposite faces
- axis through diagonally-opposite verticles
- axis through mid-points of opposite edges

Count the number of rotations about each of the axes given above. Convince yourself that you ought to get 24 such rotations.

(iii) Can you guess which group of order 24 this is?

(i) There are six (6) faces, eight (8) vertices, and twelve (12) edges.

(ii) We ought to consider the "identity rotation" — not rotating at all! This is $\boxed{1}$ rotation.

Consider an axis through diagonally-opposite vertices. We can rotate through an angle of $2\pi/3$ radians or $4\pi/3$ radians. So, there are four pairs of vertices (hence, four axes) and two angles to rotate through, so there are $\boxed{8}$ rotations possible.

Consider an axis through the mid-points of pairs of opposite edges. We can rotate through an angle of π radians. So, there are six pairs of opposite edges (hence, six axes) and only one angle to rotate through, so there are $\boxed{6}$ rotations possible.

Consider an axis through the centers of opposite faces. There are three pairs of opposite faces (hence, three axes) and three possible rotations to make, so there are $\boxed{9}$ rotations possible.

So, there are a total of $\boxed{1+8+6+9=24}$ distinct rotations of the cube possible.

(iii) This is the group S_4, which has order $|S_4| = 4! = 24$.