ES.S20: Group Theory

Worksheet #2

Problem #1

Remember: A *simple group* is a group which has no proper normal subgroups.

Show that, if G is a simple abelian group, then G must be cyclic of prime order. Give some examples.

Problem #2

(i) Describe S_3 in the form $S_3 = NK$ where $N \triangleleft S_3$ and $N \cap K = 1$. Hint: Use the result of Worksheet #1, Problem #10.

(ii) If G is a group of order 6, show that G is either cyclic of order 6 or isomorphic to S_3.

Problem #3

Consider A_4, which is the subgroup of S_4 containing all the even permutations of S_4.

(i) Show $A_4 \triangleleft S_4$.

(ii) How many cosets does A_4 have in S_4?

(iii) Explain the formula $S_4/A_4 \cong \{1, -1\}$.

(iv) Give the order of S_4 and A_4.

Problem #4

Refer to the multiplication table of A_4.

(i) Find an abelian subgroup of A_4 with four elements.

(ii) Is the subgroup you found in **(i)** cyclic?

(iii) What are the orders of the elements in the group of order 4?

(iv) Show that the subgroup of order 4 is normal in A_4.

(v) Can you find the Sylow 2-subgroups of S_4?

Problem #5

(i) Suppose that A is an $n \times n$ matrix with real entries and that P is a non-singular matrix. Show that the relation $\sim$, given by $B \sim A$ if and only if $B = P^{-1}AP$, is an equivalence relation on the real $n \times n$ matrices. A and B can then be called *similar* matrices.

(ii) A straightforward matrix which lies in each equivalence class is called a *canonical form*. Can you say anything about this using the knowledge you have at the moment?

Problem #6

This problem examines a group of order 6.

(i) Consider the rotations of the triangle $\triangle ABC$, which is an equilateral triangle. Show that we have 3 such rotations. Show that there is also a reflection of the triangle.

(ii) These symmetries of the triangle form a group called the *dihedral group* D_3 of order 6. Find a subgroup of D_3 of order 3. Following this, find a subgroup D_3 of order 2.

(iii) Show D_3 is isomorphic to S_3.

Problem #7

Problem #7 work with *Lagrange's theorem* — and the fact that the converse of Lagrange's theorem is not true in general.

(i) Show that the converse of Lagrange's theorem is true if G is cyclic. Hence show that the converse of Lagrange's theorem holds for groups of prime order.

(ii) Does the converse of Lagrange's theorem hold in groups of order p^2, where p is prime?

(iii) Does the converse of Lagrange's theorem hold for groups G of order $2p$, where p is prime? Hint: Use the result of Worksheet #2, Problem #6.

Problem #8

The *quaternions*, denoted by Q, are a group of order 8 with two generators a, b that satisfy the following three relations.

$$a^4 = 1 \qquad b^2 = a^2 \qquad b^{-1}ab = a^{-1}$$

Let G be the multiplicative group of all 2×2 non-singular complex matrices, and let H be the subgroup of G generated by

$$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Prove that $H \cong Q$.

A_4 Multiplication Table

	1	**t**	**u**	**v**	**a**	**b**	**c**	**d**	**p**	**q**	**r**	**s**
1	1	t	u	v	a	b	c	d	p	q	r	s
t	t	1	v	u	b	a	d	c	q	p	s	r
u	u	v	1	t	c	d	a	b	r	s	p	q
v	v	u	t	1	d	c	b	a	s	r	q	p
a	a	c	d	b	p	r	s	q	1	u	v	t
b	b	d	c	a	q	s	r	p	t	v	u	1
c	c	a	b	d	r	p	q	s	u	1	t	v
d	d	b	a	c	s	q	p	r	v	t	1	u
p	p	s	q	r	1	v	t	u	a	d	b	c
q	q	r	p	s	t	u	1	v	b	c	a	d
r	r	q	s	p	u	t	v	1	c	b	d	a
s	s	p	r	q	v	1	u	t	d	a	c	b

1 denotes the identity permutation.

$t = (1\,2)\,(3\,4)$ $u = (1\,3)\,(2\,4)$ $v = (1\,4)\,(2\,3)$

$a = (1\,2\,3)$ $b = (1\,3\,4)$ $c = (2\,4\,3)$ $d = (1\,4\,2)$

$p = (1\,3\,2)$ $q = (2\,3\,4)$ $r = (1\,2\,4)$ $s = (1\,4\,3)$