ES.S20: Group Theory

Worksheet #2

Problem #1

Remember: A *simple group* is a group which has no proper normal subgroups.

Show that, if G is a simple abelian group, then G must be cyclic of prime order. Give some examples.

If G is abelian, then $ab = ba$ for all $a, b \in G$. It is then true that any subgroup N of G is normal, for if $a \in N$ then $b^{-1}ab = a$ for all $b \in G$, so every conjugate of a lies in N. Hence if G is simple and N is a subgroup of G, then $N = 1$ or $N = G$. So, $\langle a^r \,|\, a \in G$ and $a \neq 1$ and $r \in \mathbb{Z}\rangle$, which is the cyclic group generated by a, must equal G.

So, $G = \langle a^r \,|\, r = 0, 1, 2, \ldots, n-1$ where $n = |G|\rangle$ is cyclic generated by a. We will now show that n is prime. If $n = kp$ and $k \neq 1$, then $\langle a^p\rangle$ is a proper subgroup of G of order k and will be normal by the above reasoning. This is not possible since G is simple, so $k = 1$ and $G = \langle a \,|\, a^p = 1\rangle$ is cyclic of prime order. Examples: $\langle a \,|\, a^3 = 1\rangle$ and $\langle a \,|\, a^5 = 1\rangle$ are simple and abelian.

Now, a counter-example — but not really, since 6 is not prime. If $G = \langle a \,|\, a^6 = 1\rangle$, then G has a proper normal subgroup. $H = \langle a^2 \,|\, 1, a^2, a^4\rangle$ is a proper normal subgroup of G.

Problem #2

(i) Describe S_3 in the form $S_3 = NK$ where $N \triangleleft S_3$ and $N \cap K = 1$. Hint: Use the result of Worksheet #1, Problem #10.

(ii) If G is a group of order 6, show that G is either cyclic of order 6 or isomorphic to S_3.

(i) $S_3 = NK$ where $N = \{1, (1\,2\,3), (1\,3\,2)\}$ and $K = \langle 1, (1\,2)\rangle$. Check that one gets all of S_3 from products nk where $n \in N$ and $k \in K$. N is of index 2 in S_3 so N is normal. Alternatively, note that all conjugates of a 3-cycle are 3-cycles and they all lie in N.

(ii) If G has an element of order 6 then G is cyclic of order 6. If $a \in G$ does not have order 6 then $o(a)$ is 2 or 3. Not all elements can be of order 2 since the group would then be of order of a power of 2. (Similar reasoning applies for a group of order 3.) Hence associate a of order 3 with $(1\,2\,3)$ and b of order 2 with $(1\,2)$ and we get a one-to-one correspondence which preserves the group structure.

Problem #3

Consider A_4, which is the subgroup of S_4 containing all the even permutations of S_4.

(i) Show $A_4 \lhd S_4$.

(ii) How many cosets does A_4 have in S_4?

(iii) Explain the formula $S_4/A_4 \cong \{1, -1\}$.

(iv) Give the order of S_4 and A_4.

In S_4, half the permutations are even and half the permutations are odd.

Define $\theta : S_4 \mapsto \{1, -1\}$. If $a \in S_4$, then define $a\theta = 1$ if a is even, and define $a\theta = -1$ is a is odd. This is an *onto map.*

$\ker(\theta) = A_4$ so $A_4 \lhd S_4$. There are two cosets of A_4 in S_4 since $|S_4|/|A_4| = 2$.

$|S_4| = 4! = 24$ by an obvious argument. Hence $|A_4| = \frac{1}{2} \cdot 4! = 12$.

Problem #4

Refer to the multiplication table of A_4.

(i) Find an abelian subgroup of A_4 with four elements.

(ii) Is the subgroup you found in **(i)** cyclic?

(iii) What are the orders of the elements in the group of order 4?

(iv) Show that the subgroup of order 4 is normal in A_4.

(v) Can you find the Sylow 2-subgroups of S_4?

(i) By inspection of the table, $V_4 = \{1, t, u, v\}$.

(ii) No, all non-trivial elements of V_4 are of order 2 — again, by inspection.

(iii) Order 2. See the argument above.

(iv) Use the fact that V_4 contains the only elements of order 2 in A_4.

(v) $|S_4| = 24 = 2^3 \cdot 3$. So, Sylow 2-subgroups have order 8. If H is any 2-subgroup in S_4, then since V_4 is a normal 2-subgroup in S_4 then HV_4 is a 2-group. Hence V_4 is contained in any Sylow 2-subgroup, since Sylow 2-subgroups are maximal Sylow 2-groups in S_4. Take H to be the cyclic group of order 4. Then $H \not\leq V_4$. Hence HV_4 is a 2-group whose order is greater than 4. By Lagrange's theorem, it must have order 8 and be a Sylow subgroup.

A cyclic subgroup of order 4 is generated by a 4-cycle. There are six 4-cycles. There occur in inverse pairs. So, there are three cyclic subgroups of order 4. Hence there are three Sylow 2-subgroups: $\{V_4, (1\,2\,3\,4)\}$, $\{V_4, (1\,2\,4\,3)\}$, and $\{V_4, (1\,3\,2\,4)\}$.

Problem #5

(i) Suppose that A is an $n \times n$ matrix with real entries and that P is a non-singular matrix. Show that the relation $\sim$, given by $B \sim A$ if and only if $B = P^{-1}AP$, is an equivalence relation on the real $n \times n$ matrices. A and B can then be called *similar* matrices.

(ii) A straightforward matrix which lies in each equivalence class is called a *canonical form*. Can you say anything about this using the knowledge you have at the moment?

(i) Note that $B \sim A$ if and only if $B = P^{-1}AP$ for some non-singular P.

$A \sim A$ since $A = I^{-1}AI$.

If $B \sim A$, then $A \sim B$ since $B = P^{-1}AP \implies PBP^{-1} = A$. So, $A = \left(P^{-1}\right)^{-1} BP^{-1}$ and thus $A \sim B$.

If $B \sim A$ and $A \sim C$, then $B \sim C$ since $B = P^{-1}AP$ and $A = Q^{-1}CQ \implies B = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP)$.

If we allow $B \sim A$ to denote $B = P^{-1}AP$, then Λ, a diagonal matrix, is a canonical form for matrices which are diagonalizable. (Look up what happens for a real-symmetric matrix!)

(ii) Let's restrict our attention to the set of all diagonalizable matrices over $\mathbb{C}$. So, $B \sim A$ for A, B in the set means that $B = P^{-1}AP$ if and only if $B \sim A$. Then we can find the diagonal form Λ as a member of each equivalence class.

The most general statement for the set of matrices over $\mathbb{C}$ which are all of size $n \times n$ is that we can find the so-called *Jordan canonical form* in each class. There is one and only one matrix of this shape in each class. You can (perhaps) learn more about this in 18.06.

Problem #6

This problem examines a group of order 6.

(i) Consider the rotations of the triangle $\triangle ABC$, which is an equilateral triangle. Show that we have 3 such rotations. Show that there is also a reflection of the triangle.

(ii) These symmetries of the triangle form a group called the *dihedral group* D_3 of order 6. Find a subgroup of D_3 of order 3. Following this, find a subgroup D_3 of order 2.

(iii) Show D_3 is isomorphic to S_3.

$C \to A$, $A \to B$, and $B \to C$ describe a rotation through $2\pi/3$ radians. Call this rotation θ.

Then $1, \theta, \theta^2$ form a subgroup of order 3, as $\theta^3 = 1$.

Let ϕ denote a reflection. Then $\phi^2 = 1$.

$D_3 = \{1, \phi, \theta, \theta^2, \phi\theta, \phi\theta^2\} \cong S_3$.

$\phi \leftrightarrow (1\,2)$ and $\theta \leftrightarrow (1\,2\,3)$.

$D_3 \cong S_3$.

Problem #7

Problem #7 work with *Lagrange's theorem* — and the fact that the converse of Lagrange's theorem is not true in general.

(i) Show that the converse of Lagrange's theorem is true if G is cyclic. Hence show that the converse of Lagrange's theorem holds for groups of prime order.

(ii) Does the converse of Lagrange's theorem hold in groups of order p^2, where p is prime?

(iii) Does the converse of Lagrange's theorem hold for groups G of order $2p$, where p is prime? Hint: Use the result of Worksheet #2, Problem #6.

(i) If $G = \langle a \,|\, a^n = 1 \rangle$, then if $m \,|\, n$ for $n = km$, $\langle a^k \,|\, a^{km} = 1 \rangle$ is a subgroup of order m for any divisor m of n.

(ii) $|G| = p^2$. Take $a \in G$, where $a \neq 1$. $o(a) \,|\, p^2$ per Lagrange's theorem, so either **(#1)** $o(a) = p^2$ and we have an element of order p, or **(#2)** $o(a) = p$ for all $a \in G$, in which case the converse of Lagrange's theorem holds: That is, for all proper divisors of p^2 there is an element of that order in G.

(iii) $|G| = 2p$, so G is isomorphic to the dihedral group D_p.

$$D_p = \langle a, b \,|\, b^2 = 1\, a^p = 1, b^{-1}ab = a^{-1} \rangle$$

We can see that $\langle b \rangle$ is of order 2 and $\langle a \rangle$ is of order p, so the converse of Lagrange's theorem holds.

Problem #8

The *quaternions*, denoted by Q, are a group of order 8 with two generators a, b that satisfy the following three relations.

$$a^4 = 1 \qquad b^2 = a^2 \qquad b^{-1}ab = a^{-1}$$

Let G be the multiplicative group of all 2×2 non-singular complex matrices, and let H be the subgroup of G generated by

$$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Prove that $H \cong Q$.

$$Q = \{a, a^2, a^3, 1, b, ba^2, ba, ab\}$$

Check that θ defined by $a\theta = A$ is a group isomorphism.

A quaternion can also be written as a linear combination $H = a1 + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$. Hamilton's rules hold.

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ik = j$$

The quaternions (± 1, $\pm i$, $\pm j$, and $\pm k$) form an abelian group of order 8.

A_4 Multiplication Table

	1	**t**	**u**	**v**	**a**	**b**	**c**	**d**	**p**	**q**	**r**	**s**
1	1	t	u	v	a	b	c	d	p	q	r	s
t	t	1	v	u	b	a	d	c	q	p	s	r
u	u	v	1	t	c	d	a	b	r	s	p	q
v	v	u	t	1	d	c	b	a	s	r	q	p
a	a	c	d	b	p	r	s	q	1	u	v	t
b	b	d	c	a	q	s	r	p	t	v	u	1
c	c	a	b	d	r	p	q	s	u	1	t	v
d	d	b	a	c	s	q	p	r	v	t	1	u
p	p	s	q	r	1	v	t	u	a	d	b	c
q	q	r	p	s	t	u	1	v	b	c	a	d
r	r	q	s	p	u	t	v	1	c	b	d	a
s	s	p	r	q	v	1	u	t	d	a	c	b

1 denotes the identity permutation.

$t = (1\,2)\,(3\,4)$ $\quad$ $u = (1\,3)\,(2\,4)$ $\quad$ $v = (1\,4)\,(2\,3)$

$a = (1\,2\,3)$ $\quad$ $b = (1\,3\,4)$ $\quad$ $c = (2\,4\,3)$ $\quad$ $d = (1\,4\,2)$

$p = (1\,3\,2)$ $\quad$ $q = (2\,3\,4)$ $\quad$ $r = (1\,2\,4)$ $\quad$ $s = (1\,4\,3)$