ES.S20: Group Theory

Worksheet #1

Problem #1

Let $\mathbb{R}$ denote the real numbers, and let n be a positive integer. Write $M_n(\mathbb{R})$ for the set of $n \times n$ matrices with entries from $\mathbb{R}$, and write $GL_n(\mathbb{R})$ for the set of matrices in $M_n(\mathbb{R})$ which are invertible. (That is, they all have an inverse.) Convince yourself by taking small values of n such that:

(i) $M_n(\mathbb{R})$ is a group under the operation of $+$ (that is, the addition of matrices);

(ii) $GL_n(\mathbb{R})$ is a group under matrix multiplication. $GL_n(\mathbb{R})$ is called the *general linear group*.

Problem #2

In $GL_n(\mathbb{R})$, we can look at $SL_n(\mathbb{R})$ where $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\}$. This is known as the n-dimensional *special linear group* over $\mathbb{R}$.

(i) Show that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

(ii) Show that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

Problem #3

The orthogonal group is defined by

$$O(n) = O_n(\mathbb{R}) = \{U \in M_n(\mathbb{R}) \,|\, UU^\top = I_n\}$$

where I_n is the $n \times n$ identity matrix. For example,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$U^\top$ denotes the transpose matrix of U. So, if $U = (u_{ij})$, then $U^\top = (b_{ij})$ where $b_{ij} = u_{ji}$.

(i) Show that $O(n)$ is a group. $O(n)$ is known as the n-dimensional *real orthogonal group.*

(ii) Optional: Show that the individual rows of an orthogonal matrix form an orthonormal set of vectors. That is, each column is of unit length and is orthogonal to every other column. (This will be dealt with more extensively in, e.g., 18.06.)

Problem #4

If we restrict our attention to those elements U of $O(n)$ satisfying $\det(U) = 1$, we have what is called the *special orthogonal group.*

$$SO(n) = SO_n(\mathbb{R}) \leq O(n) = \{U \in O(n) \,|\, \det(U) = 1\}$$

(i) Show that $SO(n)$ is a subgroup of $O(n)$.

(ii) Show that $SO(n)$ has index 2.

Orthogonal matrices are important because of their connection with rotations and reflections of Euclidean n-space. They arise in mechanics, physics, and chemistry, and as well as in linear algebra and geometry.

Problem #5

(i) Convince yourself (by looking at examples) that any permutation in S_n can be written as a product of disjoint cycles. A cycle of length 1 is called a *fixed point*. A cycle of length 2 is called a *transposition*.

(ii) Show the following by calculation.

$$(\alpha_1\alpha_2 \cdots \alpha_{m-1}\alpha_m) = (\alpha_1\alpha_2)(\alpha_1\alpha_3)\cdots(\alpha_1\alpha_m)$$

(iii) Use the result of **(ii)** to show that every permutation in S_n is a product of transpositions.

Problem #6

Remember: We say that $(1\,2\,3)\,(4\,5\,6\,7)$ has cycle type $1^3\,3^1\,4^1$ in S_{10}.

(i) Describe

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 3 & 6 & 2 & 5 & 4 & 9 & 7 & 8 \end{pmatrix}$$

in cycle notation. Give the cycle type.

(ii) $\Pi_1 = (1\,2\,3\,4)\,(5\,6\,7)\,(8\,9)$ in S_{10}, and $\Pi_2 = (1\,10)\,(2\,5\,9)\,(3\,4\,6\,7)$ in S_{10}. Calculate $\Pi_1\Pi_2$, and express the result in cycle notation.

Problem #7

Convince yourself that the order of a permutation is the least common multiple of the lengths of the disjoint cycles. Give some examples in S_5 and S_6.

Problem #8

(i) In S_3, find 2 elements of order 3, and 3 elements of order 2.

(ii) Consider the clockwise rotation x through $1/6^{\text{th}}$ of a revolution of a six-spoked wheel. The element x and its powers form a group. Find the order of x. Are the two groups in **(i)** and **(ii)** essentially the same?

Problem #9

Show that there is a normal subgroup in S_3 of order 3. Give the cosets of the subgroup in S_3.

Problem #10

The subgroup of S_3 containing all the even permutations, denoted by A_3, is called the *alternating group*. Show that A_3 is normal in S_3. Can you generalize this to S_n and A_n for $n > 3$?