ES.S20: Group Theory

Worksheet #1

Problem #1

Let $\mathbb{R}$ denote the real numbers, and let n be a positive integer. Write $M_n(\mathbb{R})$ for the set of $n \times n$ matrices with entries from $\mathbb{R}$, and write $GL_n(\mathbb{R})$ for the set of matrices in $M_n(\mathbb{R})$ which are invertible. (That is, they all have an inverse.) Convince yourself by taking small values of n such that:

(i) $M_n(\mathbb{R})$ is a group under the operation of $+$ (that is, the addition of matrices);

(ii) $GL_n(\mathbb{R})$ is a group under matrix multiplication. $GL_n(\mathbb{R})$ is called the *general linear group.*

(i) Define $M_2(\mathbb{R})$ as follows.

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$$

$M_2(\mathbb{R})$ is closed under addition.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$a+e$, $b+f$, $c+g$, and $d+h$ are all real numbers, so the sum of two matrices is again in $M_2(\mathbb{R})$.

The matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is the identity element for $M_2(\mathbb{R})$ under addition:

$$A + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + A = A.$$

The matrix

$$\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

is the inverse of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under addition.

(ii) $GL_2(\mathbb{R})$ is a group under multiplication. Closure under multiplication comes from the properties of $\mathbb{R}$. We have associative multiplication in $GL_2(\mathbb{R})$ because multiplication in $\mathbb{R}$ is associative. The matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the identity element under multiplication. By definition, if $A \in GL_2(\mathbb{R})$, then A^{-1} exists. So, we have a group.

Problem #2

In $GL_n(\mathbb{R})$, we can look at $SL_n(\mathbb{R})$ where $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \,|\, \det(A) = 1\}$. This is known as the n-dimensional *special linear group* over $\mathbb{R}$.

(i) Show that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

(ii) Show that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

Set $n = 2$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \text{ and } \det(A) = 1$$

If $A, B \in SL_2(\mathbb{R})$, then $\det(A) = 1$ and $\det(B) = 1$.

Consider $\det(AB) = \det(A)\det(B) = 1$. If $A \in SL_2(\mathbb{R})$, then $\det(A) = 1$.

$$\det(AA^{-1}) = \det(I_2) = 1$$

$$\det(A)\det(A^{-1}) = 1 \implies \det(A^{-1}) = \frac{1}{\det(A)} = 1 \text{ and } A^{-1} \in SL_2(\mathbb{R})$$

Problem #3

The orthogonal group is defined by

$$O(n) = O_n(\mathbb{R}) = \{U \in M_n(\mathbb{R}) \mid UU^\top = I_n\}$$

where I_n is the $n \times n$ identity matrix. For example,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$U^\top$ denotes the transpose matrix of U. So, if $U = (u_{ij})$, then $U^\top = (b_{ij})$ where $b_{ij} = u_{ji}$.

(i) Show that $O(n)$ is a group. $O(n)$ is known as the n-dimensional *real orthogonal group.*

(ii) Optional: Show that the individual rows of an orthogonal matrix form an orthonormal set of vectors. That is, each column is of unit length and is orthogonal to every other column. (This will be dealt with more extensively in, e.g., 18.06.)

It's sufficient for us to convince ourselves that $O_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$.

$$U \in O_2(\mathbb{R}) \implies U^\top = U^{-1}$$

So, if U_1 and U_2 are two elements of $O_2(\mathbb{R})$, then

$$(U_1U_2)^{-1} = U_2^{-1}U_1^{-1} = U_2^\top U_1^\top$$

by definitions of $O_2(\mathbb{R})$.

$$U_2^\top U_1^\top = (U_1U_2)^\top$$

by properties of multiplication of transposed matrices.

An matrix A that satisfies

$$A^\top = A^{-1}$$

is said to be *orthogonal.* If we take the transpose of the equation above, we get

$$(A^{-1})^\top = (A^\top)^\top = A$$

which says that the transpose of A^{-1} is the inverse of A^{-1} — which is A. Hence A^{-1} is orthogonal as well.

Problem #4

If we restrict our attention to those elements U of $O(n)$ satisfying $\det(U) = 1$, we have what is called the *special orthogonal group.*

$$SO(n) = SO_n(\mathbb{R}) \leq O(n) = \{U \in O(n) \,|\, \det(U) = 1\}$$

(i) Show that $SO(n)$ is a subgroup of $O(n)$.

(ii) Show that $SO(n)$ has index 2.

Orthogonal matrices are important because of their connection with rotations and reflections of Euclidean n-space. They arise in mechanics, physics, and chemistry, and as well as in linear algebra and geometry.

Note that, if $U \in O(n)$, then $UU^\top = I_n$, so $\det(U)\det(U^\top) = \det(I_n) = 1$. So, $\left(\det(U)\right)^2 = 1$ since $\det(U^\top) = \det(U)$. Hence $\det(U) = \pm 1$.

We can check that $SO(n)$ is a subgroup of $O(n)$. If $A, B \in SO(n)$, then $\det(A) = 1$ and $\det(B) = 1$, so $\det(AB) = \det(A)\det(B) = 1$ and $AA^{-1} = I$ yield $\det(A^{-1}) = 1/\det(A) = 1$.

$SO(n)$ has index 2 since there are matrices A such that $\det(A) = -1$. These are in the coset of $SO(n)$ in $O(n)$.

Problem #5

(i) Convince yourself (by looking at examples) that any permutation in S_n can be written as a product of disjoint cycles. A cycle of length 1 is called a *fixed point.* A cycle of length 2 is called a *transposition.*

(ii) Show the following by calculation.

$$(\alpha_1\alpha_2 \cdots \alpha_{m-1}\alpha_m) = (\alpha_1\alpha_2)(\alpha_1\alpha_3)\cdots(\alpha_1\alpha_m)$$

(iii) Use the result of **(ii)** to show that every permutation in S_n is a product of transpositions.

(i) Take an element in $\{1, 2, \ldots, n\}$ and permutation θ of S_n. Without loss of generality, take the element to be 1. Then 1θ, $1\theta^2$, ... are in a cycle. Close the cycle when $1\theta^r$ at the first such r. Continue in this way to express the effect of θ as the product of disjoint cycles.

(ii) We note that $\alpha_1\left(\alpha_1\alpha_2\cdots\alpha_{m-1}\alpha_m\right) = \alpha_2$, $\alpha_2\left(\alpha_1\alpha_2\cdots\alpha_m\right) = \alpha_3$, and so on.

Consider the product $(\alpha_1\alpha_2)(\alpha_1\alpha_3)\cdots(\alpha_1\alpha_m)$ on the right side of the equality. Then α_1 under the effect of $(\alpha_1\alpha_2)(\alpha_1\alpha_3)\cdots(\alpha_1\alpha_m)$ is α_2, α_2 is α_3, and so on.

(iii) Every cycle is a product of transpositions by **(ii)**. Using **(i)**, this means that every permutation is a product of transpositions.

Problem #6

Remember: We say that $(1\,2\,3)\,(4\,5\,6\,7)$ has cycle type $1^3\,3^1\,4^1$ in S_{10}.

(i) Describe

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 3 & 6 & 2 & 5 & 4 & 9 & 7 & 8 \end{pmatrix}$$

in cycle notation. Give the cycle type.

(ii) $\Pi_1 = (1\,2\,3\,4)\,(5\,6\,7)\,(8\,9)$ in S_{10}, and $\Pi_2 = (1\,10)\,(2\,5\,9)\,(3\,4\,6\,7)$ in S_{10}. Calculate $\Pi_1\Pi_2$, and express the result in cycle notation.

(i) $(2\,3\,6\,4)\,(7\,9\,5)\,(0)\,(1)\,(5)$ is the permutation given expressed in cycle notation. So, $1^3\,3^1\,4^1$ is the cycle type.

(ii) Calculate the composite of $\Pi_1\Pi_2$. For example, $1(\Pi_1\Pi_2) = 5$, $5(\Pi_1\Pi_2) = 7$, $2(\Pi_1\Pi_2) = 4$, $3(\Pi_1\Pi_2) = 6$, $4(\Pi_1\Pi_2) = 10$, $5(\Pi_1\Pi_2) = 7$, $6(\Pi_1\Pi_2) = 3$, $7(\Pi_1\Pi_2) = 9$, $8(\Pi_1\Pi_2) = 2$, $9(\Pi_1\Pi_2) = 8$, and $10(\Pi_1\Pi_2) = 1$.

We find that $\Pi_1\Pi_2 = (1\,5\,7\,9\,8\,2\,4\,10)\,(3\,6)$. So, we have an 8-cycle and 2-cycle. $8^1 2^1$ is the cycle type.

Problem #7

Convince yourself that the order of a permutation is the least common multiple of the lengths of the disjoint cycles. Give some examples in S_5 and S_6.

The order of a cycle is the length of the cycle. We raise Π to the m^{th} power where $\Pi = \Pi_1\Pi_2$ and m is the least common multiple of k and s, where k is the length of Π_1 and s is the length of Π_2.

If $\Pi^m = L$, then we obviously need both k and s to divide m, so m is the least common multiple of Π_1 and Π_2.

In S_5, $(1\,2\,3)\,(4\,5)$ is of order 6.

In S_3, $(1\,4\,2)\,(3\,5\,6)$ is of order 3.

Problem #8

(i) In S_3, find 2 elements of order 3, and 3 elements of order 2.

(ii) Consider the clockwise rotation x through $1/6^{\text{th}}$ of a revolution of a six-spoked wheel. The element x and its powers form a group. Find the order of x. Are the two groups in **(i)** and **(ii)** essentially the same?

(i) $S_3 = \{1, (1\,2\,3), (1\,3\,2), (1\,2), (1\,3), (2\,3)\}$

The two elements of order 3 are $(1\,2\,3)$ and $(1\,3\,2)$.

The three elements of order 2 are $(1\,2)$, $(1\,3)$, and $(2\,3)$.

(ii) The order of x is 6 since $x^6 = 1$ brings the wheel back to its original position. No, the two groups of order 6 are different. In **(ii)**, we have an element of order 6. In **(i)**, there is no element of order 6.

Problem #9

Show that there is a normal subgroup in S_3 of order 3. Give the cosets of the subgroup in S_3.

$A_3 = \{1, (1\,2\,3), (1\,3\,2)\}$ is of order 3 in S_3 and is normal. Either check directly that $(1\,2)\,(1\,2\,3)\,(1\,2)$ is in A_3 ...or note that $|S_3|/|A_3| = 2$ so there just two cosets of A_3 in S_3, so it must be normal. Cosets of A_3 in S_3 are A_3 and $A_3(1\,2) = \{(1\,2), (1\,3), (2\,3)\}$.

Problem #10

The subgroup of S_3 containing all the even permutations, denoted by A_3, is called the *alternating group*. Show that A_3 is normal in S_3. Can you generalize this to S_n and A_n for $n > 3$?

A_3 is normal in S_3 per **Problem #9**. Yes, this generalizes to S_n and A_n.

$$S_n = A_n \cup A_n(i\,j) \text{ where } A_n(i\,j) = \text{ all the odd permutations}$$

There are two cosets of A_n in S_n.

$$\begin{aligned} A_n &= \text{ all the even permutations} \\ A_n(i\,j) &= \text{ all the odd permutations} \end{aligned}$$